1. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$ with the property that any two members of $\mathcal{F}$ have non-empty intersection. Prove that $|\mathcal{F}| \leq 2^{n-1}$. Provide an example of such a family $\mathcal{F}$ of size exactly $2^{n-1}$.

2. Let $k \leq n/2$. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$ with the following properties:
   - every member of $\mathcal{F}$ has size at most $k$,
   - no member of $\mathcal{F}$ is contained in another member of $\mathcal{F}$.

   Prove that $|\mathcal{F}| \leq \binom{n}{k}$. Provide an example of such a family $\mathcal{F}$ of size exactly $\binom{n}{k}$.
   
   Hint: Adapt the proof of Sperner’s theorem.

3. Let $\mathcal{F}$ be a family of sets with the following properties:
   - every member of $\mathcal{F}$ has size exactly $k$,
   - for any three distinct sets $A, B, C \in \mathcal{F}$, we have $A \cap B \not\subset C$.

   Prove that $|\mathcal{F}| \leq 1 + \binom{k}{\lfloor k/2 \rfloor}$. Provide an example of such a family $\mathcal{F}$ of size exactly $1 + \binom{k}{\lfloor k/2 \rfloor}$.
   
   Hint: Fix a set $B \in \mathcal{F}$ and consider the sets $A \cap B$ for all $A \in \mathcal{F} \setminus \{B\}$.

4. Let $k \geq n/2$. Let $\mathcal{F}$ be a family of $k$-element subsets of $\{1, 2, \ldots, n\}$ with the property that $A \cup B \neq \{1, 2, \ldots, n\}$ for any $A, B \in \mathcal{F}$. Prove that $|\mathcal{F}| \leq \binom{n-1}{k}$. Provide an example of such a family $\mathcal{F}$ of size exactly $\binom{n-1}{k}$.

   Note: This is an exercise on the application of Erdős-Ko-Rado Theorem (to be presented in the next lecture), which asserts that the maximum size of a family of $k$-element subsets of $\{1, 2, \ldots, n\}$ any two of which have non-empty intersection is $\binom{n}{k-1}$. It will appear again in the next problem set.

5. Using Hall’s theorem, construct a covering of $(\mathcal{P}\{1, 2, \ldots, n\}, \subset)$ (the family of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion) by $\binom{n}{\lfloor n/2 \rfloor}$ chains. Use this covering to deduce Sperner’s theorem.

   Hint: Prove that the bipartite graph of inclusions between the $k$-element subsets and the $(k+1)$-element subsets of $\{1, 2, \ldots, n\}$ has a matching of size $\min\{\binom{n}{k}, \binom{n}{k+1}\}$, for every $k \in \{0, 1, \ldots, n-1\}$.

6. * Let $n \geq k \geq 2$. We say that a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ generates a subset $A \subset \{1, 2, \ldots, n\}$ if one can write $A$ in the form of an expression using the sets in $\mathcal{F}$, the operations $\cup$ and $\cap$, and parentheses. For example, the family $\{\{1, 3\}, \{2, 3, 5\}, \{4, 5\}\}$ generates the set $\{3, 5\}$, because

   $$(\{1, 3\} \cap \{2, 3, 5\}) \cup (\{2, 3, 5\} \cap \{4, 5\}) = \{3, 5\}.$$ 

   Prove that there exists a family $\mathcal{F}$ of size $k$ of subsets of $\{1, 2, \ldots, n\}$ that generates all subsets of $\{1, 2, \ldots, n\}$ if and only if $n \leq \binom{k}{\lfloor k/2 \rfloor}$. 