1. Prove that every tree with a vertex of degree $k$ has at least $k$ leaves.

2. Prove that every acyclic graph with $n$ vertices and $n - 1$ edges is a tree.

3. Recall that a path from a vertex $u$ to a vertex $v$ in a graph is a sequence $(x_0, x_1, \ldots, x_k)$ of vertices such that $x_0 = u$, $x_k = v$, and $x_ix_{i+1}$ is an edge for $0 \leq i \leq k - 1$. Prove that a graph is a tree if and only if for any two vertices $u$ and $v$ there exists exactly one path from $u$ to $v$.

4. Consider an $n$-vertex complete graph with a selected edge $e$. Prove that the graph contains $2n^{n-3}$ spanning trees containing the edge $e$.

   **Hint:** The number of spanning trees containing the edge $e$ does not depend on the choice of $e$. Consider the sum of the numbers of spanning trees containing $e$ over all edges $e$.

5. Consider a directed graph $G$ whose vertices are sequences $(x_1, x_2, \ldots, x_{k-1})$ of zeros and ones of length $k - 1$ and whose edges are defined so that for any $x_1, x_2, \ldots, x_k \in \{0, 1\}$ there is a directed edge from $(x_1, x_2, \ldots, x_k)$ to $(x_2, \ldots, x_{k-1}, x_k)$. Prove that the graph $G$ is Eulerian. Using an Euler tour in this graph, construct a sequence $(x_1, x_2, \ldots, x_n)$ of zeros and ones of length $n = 2^k + k - 1$ with no two identical blocks of size $k$, that is, with no two identical subsequences of the form $(x_{i+1}, x_{i+1}, \ldots, x_{i+k-1})$ for $1 \leq i \leq n - 1$.

   **Note:** A sequence of zeros and ones with no two identical blocks of size $k$ cannot be longer than $2^k + k - 1$.

6. * Let $T$ be a tree with $n$ vertices. A **subtree** of $T$ is a subgraph of $T$ which is itself a tree. Prove that $T$ contains a vertex $v$ with the following property: every subtree of $T$ that does not contain $v$ has at most $n/2$ vertices.