1. Prove that for \( n, k \in \mathbb{N} \) and \( k \geq n \), the number of surjective functions \( f : \{1, 2, \ldots, k\} \rightarrow \{1, 2, \ldots, n\} \) (that is, functions whose image is the entire codomain \( \{1, 2, \ldots, n\} \)) is
\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^k.
\]

2. Prove that every graph with at least two vertices contains two vertices with equal degrees.

3. The complement of a graph \( G \) with vertex set \( V \) and edge set \( E \) is the graph \( G' \) with the same vertex set \( V \) and with edge set \( \binom{V}{2} \setminus E \), where \( \binom{V}{2} \) denotes the set of all two-element subsets of \( V \). Prove that if a graph \( G \) is not connected, then its complement \( G' \) is connected.

4. A directed graph is a pair \( G = (V, E) \), where \( V \) is a non-empty finite set of vertices and \( E \subseteq V \times V \) is a set of directed edges. For a vertex \( x \in V \) we define
   - the in-degree of \( x \) to be the number of vertices \( y \) such that \((y, x) \in E\),
   - the out-degree of \( x \) to be the number of vertices \( y \) such that \((x, y) \in E\).

An Euler tour in such a graph \( G \) is defined just like for ordinary (undirected) graphs—it is a sequence \((x_0, x_1, \ldots, x_k)\) of vertices of \( G \) with the following properties:
   - \( x_0 = x_k \),
   - \((x_i, x_{i+1}) \in E\) for \( i \in \{0, 1, \ldots, k - 1\} \),
   - every vertex of \( G \) occurs in the sequence at least once,
   - every edge of \( G \) is of the form \((x_i, x_{i+1})\) for exactly one index \( i \in \{0, 1, \ldots, k - 1\} \)
     (thus, in particular, \( k = |E| \)).

A directed graph is Eulerian if it has an Euler tour. The symmetrization of a directed graph \( G = (V, E) \) is the graph \( G' = (V, E') \) such that
\[
E' = \{ \{x, y\} \in \binom{V}{2} : (x, y) \in E \text{ or } (y, x) \in E \}.
\]

Prove that a directed graph \( G \) is Eulerian if and only if the following two conditions hold:
   - \( G \) is weakly connected, that is, the symmetrization of \( G \) is connected,
   - the out-degree of every vertex is equal to its in-degree.

5. A tournament is a directed graph which contains exactly one edge connecting any two distinct vertices \( u \) and \( v \): either \((u, v)\) or \((v, u)\). Prove that every tournament contains a path going through all the vertices, each exactly once.
6. * You are going to hire a new employee. There are \( n \) candidates who submitted their applications, and you want to select the best one. However, you are unable to decide how good a candidate is without having an interview. The candidates come for an interview one at a time, in a random order. While interviewing a candidate, you can decide whether he is better than all the candidates that have come before, but you do not know whether an even better candidate will come after. However, you must tell a candidate whether he is hired or not immediately at the end of his interview.

In order to deal with this problem, you apply the following strategy. For a carefully chosen constant \( c \in [0, 1] \), you interview the first \( \lfloor cn \rfloor \) candidates and dismiss all of them. Then, you hire the first candidate that you find better than all the ones that have come before him. What value of \( c \) maximizes the probability of selecting the best candidate in the limit as \( n \to \infty \)? What is the value of that limit probability?

The following facts may be useful for your solution:

- \( \sum_{k=1}^{n} 1/k \sim \ln n \),
- the function \( f(x) = x \ln x \) is minimized for \( x = 1/e \).