1. Let $\mathcal{F}$ be a family of $k$-element subsets of a set $X$. Prove that the elements of $X$ can be colored with $k$ colors so that at least $|\mathcal{F}| \cdot k!/k^k$ sets in $\mathcal{F}$ have exactly one element of each color.

2. Consider $2n$ sets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$, each being a $k$-element subset of some fixed set $X$. Prove that if $n < 2^{k-1}$, then there are elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that $a_i \in A_i$ and $b_i \in B_i$ for $i \in \{1, 2, \ldots, n\}$, and $a_i \neq b_j$ for $i, j \in \{1, 2, \ldots, n\}$.

Hint: Consider a random partition of $X$ into two subsets $A$ and $B$, and estimate the probability that one cannot choose elements $a_1, a_2, \ldots, a_n \in A$ and $b_1, b_2, \ldots, b_n \in B$ with the required properties.

3. For a family $\mathcal{F}$ of subsets of a set $X$, let $\tau(\mathcal{F})$ denote the minimum size of a subset of $X$ whose intersection with every set in $\mathcal{F}$ is non-empty. Let $\mathcal{F}$ be a family of $k$-element subsets of a set $X$. Prove that if $\tau(\mathcal{F}) = \ell + 1$ and $\tau(\mathcal{F} \setminus \{Y\}) = \ell$ for every $Y \in \mathcal{F}$, then $|\mathcal{F}| \leq \binom{k+\ell}{k}$.

Hint: Use Bollobás’s theorem.

4. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$ with

$$|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}.$$

Prove that there is a set $S \subset \{1, 2, \ldots, n\}$ of size $k$ such that every subset $X$ of $S$ can be written as $X = Y \cap S$ for some set $Y \in \mathcal{F}$.

Hint: Apply induction on $n$. For the induction step, consider families

$$\mathcal{F}_1 = \{Y \subset \{1, 2, \ldots, n-1\}: Y \in \mathcal{F} \text{ or } Y \cup \{n\} \in \mathcal{F}\},$$

$$\mathcal{F}_2 = \{Y \subset \{1, 2, \ldots, n-1\}: Y \in \mathcal{F} \text{ and } Y \cup \{n\} \in \mathcal{F}\}.$$

Prove that $\mathcal{F}_1$ satisfies the condition of the problem for $n-1$ and $k$, or $\mathcal{F}_2$ satisfies the condition for $n-1$ and $k-1$.

5. * Let $\mathcal{F}$ be a family of subsets of size at least 2 of a set $X$ such that every element of $X$ belongs to exactly two sets in $\mathcal{F}$. Prove that exactly one of the following two conditions holds:

- there are elements $x_1, x_2, \ldots, x_n \in X$ such that $\{x_i, x_{i+1}\} \in \mathcal{F}$ for $1 \leq i \leq n-1$, $\{x_n, x_1\} \in \mathcal{F}$, and $n$ is odd,
- the elements of $X$ can be colored with two colors so that no set in $\mathcal{F}$ is monochromatic.