1 Bipartite graphs and Hall’s Theorem

A graph \((V, E)\) is bipartite if it has a partition \(V = A \cup B\) of the vertex set into two disjoint subsets such that every edge connects a vertex in \(A\) with a vertex in \(B\). By convention, we write \((A \cup B, E)\) for a bipartite graph to denote that \(A \cup B\) is a partition as above. A set of edges \(M \subseteq E\) is a matching in a bipartite graph \((A \cup B, E)\) if every vertex is contained in at most one edge in \(M\). For a bipartite graph \(G = (A \cup B, E)\) and a set of vertices \(X \subseteq A\), let \(N_G(X)\) denote the set of neighbors of \(X\) in \(G\), that is,

\[
N_G(X) = \{b \in B : \exists a \in X : ab \in E\}.
\]

Hall’s Theorem. A bipartite graph \(G = (A \cup B, E)\) has a matching of size \(|A|\) if and only if the following condition is satisfied:

\[\text{for every set } X \subseteq A \text{ we have } |N_G(X)| \geq |X|, \quad (*)\]

Proof. Suppose there is a matching \(M\) of size \(|A|\) in \(G\). We show that the condition \((*)\) holds for \(G\). Since \(|M| = |A|\), every vertex in \(A\) is contained in some edge of \(M\). Therefore, for any set \(X \subseteq A\), there are exactly \(|X|\) edges in \(M\) having a common vertex with \(X\). The other vertex in each of these edges belongs to \(N_G(X)\). This shows that \(|N_G(X)| \geq |X|\).

The proof of the converse implication goes by induction on \(|A|\). Suppose that \(G\) satisfies the condition \((*)\), that is, for every set \(X \subseteq A\) we have \(|N_G(X)| \geq |X|\). If \(|A| = 1\), then the condition \((*)\) yields \(|N_G(A)| \geq 1\), so there is an edge containing the vertex in \(A\) and another vertex in \(B\). That edge alone forms a matching of size 1.

Now, assume \(|A| \geq 2\). We consider two cases.

Case 1. For every set \(X \subseteq A\) such that \(X \neq \emptyset\) and \(X \neq A\) we have \(|N_G(X)| > |X|\).

In this case, choose any edge \(ab \in E\). Let \(A' = A \setminus \{a\}\), \(B' = B \setminus \{b\}\), and \(E' \subseteq E\) be the set of those edges that connect a vertex in \(A'\) with a vertex in \(B'\). We claim that the condition \((*)\) holds for the graph \(G' = (A' \cup B', E')\). Indeed, for any non-empty set \(X \subseteq A'\), we have \(N_{G'}(X) = N_G(X) \setminus \{b\}\), and thus

\[|N_{G'}(X)| = |N_G(X) \setminus \{b\}| \geq |N_G(X)| - 1 \geq |X|.
\]

Therefore, we can apply the induction hypothesis to \(G'\) to obtain a matching \(M'\) of size \(|A'| = |A| - 1\) in \(G'\). The set \(M = M' \cup \{ab\}\) forms a matching of size \(|A|\) in \(G\).

Case 2. There is a set \(A' \subseteq A\) such that \(A' \neq \emptyset\), \(A' \neq A\), and \(|N_G(A')| = |A'|\).

In this case, let \(A'' = A \setminus A', B' = N_G(A'),\) and \(B'' = B \setminus B'\). In particular, we have \(|A'| = |B'||. Let \(E'' \subseteq E\) be the set of those edges that go between \(A'\) and \(B'\), and let \(E'' \subseteq E\) be the set of those edges that go between \(A''\) and \(B''\). We claim that the condition \((*)\) holds for the graphs \(G'' = (A' \cup B', E'')\) and \(G'' = (A'' \cup B'', E'\)\). It holds for \(G''\), because for any set \(X \subseteq A'\), we have \(N_{G''}(X) = N_G(X) \cap B' = N_G(X)\)), so \(|N_{G''}(X)| = |N_G(X)| \geq |X|\). It holds for \(G''\), because for any set \(X \subseteq A'',\) we have \(N_{G''}(X) = N_G(X) \setminus B' = N_G(X \cup A') \setminus B'\), so

\[|N_{G''}(X)| = |N_G(X \cup A')| - |B'| \geq |X| + |A'| - |B'| = |X| + |A'| - |B'| = |X|,
\]

where the second equality follows from the fact that \(B' \subseteq N_G(X \cup A')\). Therefore, we can apply the induction hypothesis to both \(G'\) and \(G''\) to obtain a matching \(M'\) of size \(|A'|\) in \(G'\) and a matching \(M''\) of size \(|A''|\) in \(G''\). The set \(M = M' \cup M''\) forms a matching of size \(|A'| + |A''| = |A|\) in \(G\). \(\square\)
2 Partially ordered sets and Dilworth’s Theorem

A partially ordered set is a pair \((X, \leq)\), where \(X\) is a set and \(\leq\) is a partial order relation on \(X\), that is, binary relation with the following properties:

\[\begin{align*}
\bullet \quad & a \leq a, \\
\bullet \quad & \text{if } a \leq b \text{ and } b \leq a \text{, then } a = b, \\
\bullet \quad & \text{if } a \leq b \text{ and } b \leq c \text{, then } a \leq c.
\end{align*}\]

Here, \(X\) will be always non-empty and finite. We write \(a < b\) to denote that \(a \leq b\) and \(a \neq b\).

Elements \(a, b \in X\) are comparable if \(a \leq b\) or \(b \leq a\), and otherwise they are incomparable.

A set \(C \subset X\) is a chain if it consists of pairwise comparable elements. One can always enumerate the elements of a chain \(C\) as \(x_1, x_2, \ldots, x_k\) so that \(x_1 < x_2 < \ldots < x_k\), where \(k = |C|\). A set \(A \subset X\) is an antichain if it consists of pairwise incomparable elements. A chain and an antichain can have at most one common element, because two elements cannot be simultaneously comparable and incomparable.

Example. For a set \(X\), let \(\mathcal{P}X\) denote the power set of \(X\), that is, the set of all subsets of \(X\). Then, \((\mathcal{P}X, \subset)\) is a partially ordered set. It looks as follows for \(X = \{1, 2, 3\}\):

![Diagram of partially ordered set]

In the diagram above, we use the convention that lines connecting two elements of the partially ordered set represent comparability (the lower element is “smaller” than the upper one), except that we do not draw comparabilities that follow from transitivity (the last condition of a partial order relation). In this example, \(\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\) is a chain, while \(\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}\) is an antichain.

The width of a partially ordered set \((X, \leq)\) is the maximum size of an antichain in \(X\). We say that a collection of chains \(C_1, C_2, \ldots, C_k\) covers \(X\) if \(C_1 \cup C_2 \cup \ldots \cup C_k = X\).

Dilworth’s Theorem. The width of a partially ordered set \((X, \leq)\) is equal to the minimum number of chains necessary to cover \(X\).

Proof. Since every antichain in \(X\) can have at most one common element with each chain from the cover, the width of \((X, \leq)\) cannot be greater than the number of chains covering \(X\). We prove that one can always find chains covering \(X\) whose number is equal to the width of \((X, \leq)\).

The proof goes by induction on \(|X|\). If \(|X| = 1\), then the width is 1 and \(X\) itself is a chain that covers \(X\). More generally, if \(X\) is a chain, then the width is 1 (there are no two incomparable elements), and \(X\) is a chain that covers \(X\).

Now, assume that \(|X| \geq 2\) and \(X\) is not a chain. Let \(a \in X\) be maximal, that is, such that there is no element \(b \in X\) with \(b > a\). Let \(X' = X \setminus \{a\}\), and let \(k\) be the width of...
(X′, ≤|X′)). By the induction hypothesis, X′ can be covered by k chains C1, C2, . . . , Ck. We can assume without loss of generality that the chains C1, C2, . . . , Ck are pairwise disjoint (if not, we remove some elements from some of the chains to make them pairwise disjoint). For each i ∈ {1, 2, . . . , k}, let xi be the maximal element of Ci with the property that there is an antichain in X′ of size k containing xi (that is, there is no element y ∈ Ci such that y > xi and y belongs to an antichain in X′ of size k). At least one element xi with the property above exists in each Ci, because there is some antichain of size k in X′ (witnessing the width), which must contain a common element with each of C1, C2, . . . , Ck. The elements x1, x2, . . . , xk are pairwise distinct, because they are taken from pairwise disjoint chains.

We claim that {x1, x2, . . . , xk} in an antichain. Suppose not, so that we have xi > xj for some i ̸= j. Let A be an antichain in X′ of size k containing xi, which exists by the definition of xi. The set A contains one element in each of C1, C2, . . . , Ck. Let y be the common element of A and Cj. The element y has the property that there is an antichain in X′ of size k containing y (namely A), and thus by the definition of xj, we have xj ≥ y. This together with xi > xj yields xi > y, which contradicts the assumption that A is an antichain. The contradiction shows that {x1, x2, . . . , xk} is indeed an antichain.

Now, we consider two cases.

Case 1. For some index i ∈ {1, 2, . . . , k}, we have a > xi.

Let K = {a} ∪ {y ∈ Ci : y ≤ xi}. Since a > xi, the set K is a chain in X. We have K ̸= X, because we have assumed that X is not a chain. Let X′ = X \ K. We claim that the width of (X′, ≤|X′|) is at most k − 1. Suppose not, so that there is an antichain A of size k in X′.

The set A contains one element in each of the k chains C1, C2, . . . , Ck \ K, Ck+1, . . . , Ck. Let y be the common element of A and Cj. The element y has the property that there is an antichain in X′ of size k containing y (namely A). This contradicts the choice of xi as the maximal element of Ci with this property. The contradiction shows that the width of (X′, ≤|X′|) is indeed at most k − 1. Actually, it is exactly k − 1, because {x1, x2, . . . , xk} \ {xi} is an antichain of size k − 1 in X′.

Now, we apply the induction hypothesis to (X′, ≤|X′|) and conclude that X′ can be covered by k − 1 chains. These k − 1 chains together with K are k chains covering X. Since k is the width of X′, it is also the width of X.

Case 2. For every i ∈ {1, 2, . . . , k}, the elements a and xi are incomparable.

In this case, {a, x1, x2, . . . , xk} is an antichain in X of size k + 1. We also have a covering of X by k + 1 chains {a}, C1, C2, . . . , Ck. Therefore, the width of X is k + 1.

In both cases, we have found a requested covering of X with a number of chains equal to the width of X.

3 Another proof of Hall’s Theorem

As an example of an application of Dilworth’s Theorem, we present an alternative proof of Hall’s Theorem.

Proof of Hall’s Theorem. We focus on the non-obvious implication, which is that the condition (*) implies the existence of a matching of size |A|. So suppose that the condition (*) is satisfied for G. In particular, we have |B| ≥ |A|. Consider a partially ordered set (A ∪ B, ≤) with the relation ≤ defined as follows:

\[ a ≤ b ⇔ a = b \text{ or } a ∈ A, b ∈ B \text{ and } ab ∈ E. \]

It is easy to check that this relation is indeed a partial order.

Since B is an antichain, the width of (A ∪ B, ≤) is at least |B|. We claim that it is exactly |B|. Suppose not, that there is an antichain D ⊂ A ∪ B such that |D| > |B|. It follows that
\( N_G(A \cap D) \subset B \setminus D \), because each neighbor of an element of \( A \cap D \) is comparable to that element. Therefore,

\[
|N_G(A \cap D)| \leq |B \setminus D| = |B| - |B \cap D| = |B| - |D| + |A \cap D| < |A \cap D|.
\]

This contradicts the condition (\( \ast \)). Hence the width of \((A \cup B, \leq)\) is exactly \(|B|\).

Now, we apply Dilworth’s Theorem to \((A \cup B, \leq)\) and conclude that \(A \cup B\) can be covered by \(|B|\) chains. Each of these chains contains exactly one element of \(B\) and at most one element of \(A\), so it has size either 1 or 2. Since there are \(|A| + |B|\) elements to cover, exactly \(|A|\) of these chains have size 2, that is, contain one element of \(A\) and one element of \(B\). The elements of each of these \(|A|\) chains are comparable, so they are connected by an edge in \(G\). These \(|A|\) edges form a matching in \(G\). \(\square\)