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A NEW UPPER BOUND FOR THE CARDINALITY OF 2-DISTANCE SETS IN EUCLIDEAN SPACE

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It is proved that the cardinality of a 2-distance set S in Euclidean d-dimensional space satisfies

$$\operatorname{card}(S) \leq \tfrac{1}{2}(d+1)(d+2).$$

A set S in Euclidean d-space, E^d, is called a 2-distance set if the distance between distinct points of S assumes only two values. The maximum size of such a set is 5 in E^2 (Kelly), and 6 in E^3 (Croft). Delsarte, Goethals and Seidel [1] treated the case where the points of S lie on a sphere. Their argument can be modified to obtain the bound $\operatorname{card}(S) \leq \frac{1}{2}(d+1)(d+4)$ for general 2-distance sets as was established by Larman, Rogers and Seidel [2]. Bannai and Bannai [3] showed that equality does not occur in this case. The proof of Larman, Rogers and Seidel can be modified again to obtain $\operatorname{card}(S) \leq \frac{1}{2}(d+1)(d+2)$.

Theorem. *Let S be a 2-distance set in E^d, then*

$$\operatorname{card}(S) \leq \tfrac{1}{2}(d+1)(d+2).$$

Proof. Let a and b be the distances in S. For each point s in S and $x \in E^d$ we define

$$F_s(x) = \frac{1}{a^2 b^2}(\|x-s\|^2 - a^2)(\|x-s\|^2 - b^2).$$

These functions form an independent set of functions since $F_s(t) = \delta_{s,t}$ for all $s, t \in S$. They are linear combinations of the following functions:

$$\|x\|^4; \qquad \|x\|^2 x_i; \qquad x_i x_j; \qquad x_i; \qquad 1, \quad \text{where } 1 \leq i \leq j \leq d.$$

Hence, the total number of functions F_s cannot exceed

$$1 + d + \tfrac{1}{2}d(d+1) + d + 1 = \tfrac{1}{2}(d+1)(d+4).$$

We proceed to show that in fact the set

$$\{F_s(x), x_i, 1 \mid s \in S, 1 \leq i \leq d\}$$

is linearly independent, which implies

$$\operatorname{card}(S) + d + 1 \leq \tfrac{1}{2}(d+1)(d+4),$$

and hence

$$\operatorname{card}(S) \leq \tfrac{1}{2}(d+1)(d+2).$$

Now suppose we have:

$$\sum_{s \in S} c_s F_s(x) + \sum_{i=1}^{d} c_i x_i + c = 0. \tag{1}$$

Inserting s into relation (1) we get:

$$c_s + \sum_i c_i s_i + c = 0. \tag{2}$$

Inserting ke_i into (1), where e_i is the ith column of the unit matrix, we get:

$$\frac{1}{a^2 b^2} \sum_s c_s (k^2 - 2ks_i + \|s\|^2 - a^2)(k^2 - 2ks_i + \|s\|^2 - b^2) + kc_i + c = 0. \tag{3}$$

Comparing the coefficients of k^4 and of k^3 we obtain:

$$\sum_s c_s = 0 \quad \text{and} \quad \sum_s c_s s_i = 0, \tag{4}$$

for $i = 1, \ldots, d$.

Multiply relation (2) by c_s and sum over all $s \in S$:

$$\sum_s c_s^2 + \sum_i c_i \sum_s c_s s_i + c \sum_s c_s = 0. \tag{5}$$

Now (4) and (5) yield $c_s = 0$ for all $s \in S$, whence also $c = c_i = 0$ for $i = 1, \ldots, d$. This completes the proof of the theorem.

References

[1] Ph. Delsarte, J.M. Goethals and J.J. Seidel, Spherical codes and designs, Geometrica Dedicata 6 (1977) 363–388.

[2] D.G. Larman, C.A. Rogers and J.J. Seidel, On two-distance sets in Euclidean space, Bull. London Math. Soc. 2 (1977) 261–267.

[3] E. Bannai and E. Bannai, An upper bound for the cardinality of an s-distance subset in Euclidean space (to appear in Combinatorica).

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