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SETS OF LATTICE POINTS THAT FORM NO SQUARES

by

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Consider the lattice points in a square of size $n \times n$, i.e. consider e.g. the points (i, j) in the Cartensian plane where $0 \leqq i,\ j \leqq n$. P. ERDŐS and R. GRAHAM [1] conjectured that given any positive constant c, any set of cn^2 elements from among the above lattice points contains four points that form a square the sides of which are parallel to the coordinate axes provided n is large enough, depending on c. We are unable to settle this problem here. Instead we consider the easier case of isoceles right-angled triangles whose sides are parallel to the axes; the consideration of this case was also suggested by ERDŐS and GRAHAM. First we prove the following

THEOREM 1. *There is a positive absolute constant c such that, given any positive integer n, there are $c(r_3(n))^2$ lattice points in a square of size $n \times n$ no three of which form an isoceles right-angled triangle whose sizes are parallel to the axes.*

Here $r_3(n)$ denotes the largest integer m such that there are m integers less than n no three of which form an arithmetic progression. According to a result of F. BEHREND, we have $r_3(n) > n^{1-c/\sqrt{\log n}}$.

PROOF. Let $a_1, \ldots, a_k$ $(k = r_3([n/2])$, where $[\cdot]$ stands for the integral part of the bracketed number) be a sequence of integers less than n containing no arithmetic progression of three elements, and consider the set S of those lattice points in the square $0 \leqq x,\ y \leqq n$ that are in the intersection of lines of form $y = x + 2a$; and $y = -x + 2a$; $(1 \leqq i,\ j \leqq k)$. As any two such line intersects each other in a lattice point, it is easy to see that S has at least $k^2/4$ elements. As $r_3([n/2])$ and $r_3(n)$ have the same orders of magnitude, it is clear that the cardinality of S is at least $c(r_3(n))^2$, where c is an easily computable absolute constant. We show that there are no three points belonging to S that form a right-angled isoceles triangle with sides parallel to the axes. Assume, on the contrary, that ABC is such a triangle, say $\overline{AC} = \overline{BC}$. We may assume, without loss of generality, that each of the points A, B, and C lies on different lines of form $y = x + 2a$; (the other possibility, when each of A, B, and C lies on different lines of form $y = -x + 2a$ can be dealt with similarly); say A lies on the line $y = x + 2a_r$, C on $y = x + 2a_s$, and B on $y = x + 2a_t$. Then a_r, a_s, and a_t form an arithmetic progression, which is a contradiction. The proof complete.

In the other direction, we prove the following:

THEOREM 2. *Let $c > 0$, and let S be a set of cn^2 of lattice points in a square of size $n \times n$. Then there are three points in S that form an isoceles right-angled triangle whose sides are parallel to the axes, provided n is large enough.*

PROOF. Assume that the assertion of the theorem fails, and let c_n be the largest constant such that there are $c_n n^2$ points that violate the assertion of the theorem, and put $c = \varlimsup_{n \to \infty} c_n$. Then $c > 0$ holds according to our assumption. Let n be a large integer, and let S be a set of $(c-\varepsilon)n^2$, points in a square of size $n\times n$ such that S does not contain an isoceles triangle whose sides are parallel to the axes. We are going to obtain a contradiction by showing that there is a square of size $k\times k$ that contains $(c+\varepsilon)k^2$ elements of S (i.e. there are a_1, a_2, and d such that there are $(c+\varepsilon)k^2$ elements of S of form (a_1+id, a_2+jd) with $0\leqq i, j<k$).

First we show that, almost every line $x=i$ $(0\leqq i<n)$ contains at most $(c+\sqrt{\varepsilon})n$ elements of S, i.e. the number of exceptional i's is $o(n)$. Let us consider the lines $x=i$ for $t\sqrt{n}\leqq i<(t+1)\sqrt{n}$, where $0\leqq t<\sqrt{n}$. It is enough to show that the number of exceptional i's satisfying this condition is $o(\sqrt{n})$. In fact, assume, on the contrary, that there is a positive constant η independent of n that there are $\eta\sqrt{n}$ exceptional i's with $t\sqrt{n}\leqq i<(t+1)\sqrt{n}$ for some $0\leqq t<\sqrt{n}$. The set H of exceptional i's satisfying this condition can contain an arithmetic progression $a+jd$ $(0\leqq j<k)$ of length k, where k can be arbitrarily large, depending on n; note that we use the following theorem here: an infinite sequence of integers of positive upper density contains an arbitrarily long arithmetic progression (see [2]). Now split the set of positive integers $\leqq n$ as $T_0\cup\bigcup_{0<l<n/k} T_l$, T_l is an arithmetic progression of length k with difference d and T_0 contains at most $\eta\cdot n$ elements (note that $d<\sqrt{n}$). Then there is an l so that $A_l=T_l\times\{a+jd\colon 0\leqq j<k\}$ contains at least $(c+2\varepsilon)k^2$ elements of S, since each $a+jd$ was exceptional. If k is large enough, then in view of the definition of c, the set $A_l\cap S$ contains an isoceles triangle with sides parallel to the axes, which contradicts the definition of S.

So we have shown that, loosely speaking only very few lines $x=i$ $(0\leqq i<n)$ can contain "more then the average" number of elements of S; therefore, only a few lines contain "less than the average" number of elements of S; more precisely, an easy computation gives that there are less than $2\sqrt{\varepsilon}n$ lines that contain more than $(c+\varepsilon^{1/4})n$, elements of S.

Now consider the lines $y=x+h$, where $-n\leqq h\leqq n$. If ε is small enough, say $\varepsilon<c/100$ then there is such a line with $h=h_0$ containing $cn/4$ elements of S. Omit those points which lie on a vertical line containing less than $(c-\varepsilon^{1/4})n$ elements of S; there still remain say $cn/6$ points. Let these elements of S have abcissas a_i $(0\leqq i<cn/6)$. According to a theorem cited just before, there is an arithmetic progression $a+jd$ of length k each elements of which is some a_i; and we may also assume that $d<\sqrt{n}$. Now decompose the set of integers $\leqq n$ as $T_0\cup\bigcup_{0<l<n/k} T_l$ as above (with the new parameters d and k).

The rectangle $[0, n)\times\{a+jd\colon 0\leqq j<k\}$ contains at least $(c-\sqrt{\varepsilon})nk$ points of S as each line $x=a+jd$ contains at least $(c-\sqrt{\varepsilon})n$; no square $T_l\times\{a+jd\colon 0\leqq j<k\}$ may contain more than $(c+\varepsilon)k^2$ points if k is large enough in view of the definition of c. So almost every T_l $\left(\text{i.e. except at most } \varepsilon^{1/4}\frac{n}{k}\right)$ contains at least $(c-\varepsilon^{1/4})k^2$ elements of S. We may assume that all of them do, as those T_l's which do not, may be added to T_0, which will thus have cardinality $\leqq 2\varepsilon^{1/4}n$. There is an l such that T_l contains at least $ck/8$ elements of a_i+h_0. Note that no line $y=a_i+h_0$ can contain an element with abscissa a_j for any j (so, in particular, with abscissa belonging to the arithmetic progression $a+jd$) as then (a_i, a_i+h), (a_i, a_j), (a_j, a_j+h) would be an isoceles triangle. This means that at least $ck/8$ lines of the square

$$T_l\times\{a+jd\colon 0\leqq j<k\}$$

contains no element of S which is a contradiction, provided k is large enough as we saw that almost every line of a square must contain many elements of S. (We proved this for the original square n, provided S is large enough; but these conditions hold for the above smaller square too, as it contains at least $(c-\varepsilon^{1/4})k^2$ elements of S.) The proof is complete.

REFERENCES

[1] ERDÖS, P.: *Problems and results on Combinatorial Number Theory*, A Survey of Combinatorial theory, North Holland Publishing Company 1973, (117—137)

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