Advanced Discrete Mathematics 2013 – Problem Set 8 – Solutions

1. *Show that if G has ≥ 4 vertices and every two of its vertices have exactly one common neighbor, but there is no vertex adjacent to all the others, then G must be regular.*
(This is part of the proof of the windmill theorem.)

   Please draw a picture when reading this solution. Note that the question is a bit weird, because this is part of a proof by contradiction, so a graph satisfying these condition does not actually exist. Still, I think it is a nice exercise.
   We write $f(u, v)$ for the common neighbor of two vertices. Take two non-adjacent vertices u, v and their unique common neighbor $w = f(u, v)$. Then because $uv \notin E$, $f(u, w)$ is not v, and $f(w, v)$ is not u. Also $f(u, w) \neq f(w, v)$ because that would give a C_4. So $uwf(u, w)$ and $wvf(w, v)$ form two triangles sharing only the vertex w.
   For any other neighbor t of u, $f(t, v)$ is not one of the 5 previous vertices, because in each case we would get a C_4. So we can pair up t with a neighbor of v. We can repeat this for any other $t' \in N(u)$, and then also $f(t', v) \neq f(t, v)$ and $f(t', v) \neq t$, because that would give a C_4. Repeating this we can pair off the remaining neighbors of u and v, hence $d(u) - 2 = d(v) - 2$, showing that $d(u) = d(v)$. So any 2 non-adjacent vertices have the same degree. But any vertex other than w is non-adjacent to at least one of u, v, so these all have the same degree k. And by the assumption at the start, w is not a neighbor of all vertices, so it will also have degree k.

2. *Find two non-isomorphic graphs that have the same spectrum, with one connected and the other one not connected. Conclude that connectedness cannot always be determined from the spectrum.*

   The graphs below both have spectrum $(2)^1(0)^3(-2)^1$.

3. (a) *Show that if t is the number of triangles K_3 in a graph, then*

$$6t = \sum_{i=1}^{n} \lambda_i^3.$$

   By a lemma in the notes, this sum equals the number of closed walks of length 3. This counts every triangle 6 times, once for each starting point and direction of the walk.

(b) *Use this to (re)prove that if a graph has $> n^2/4$ edges, then it contains a K_3.* **Hint:** Use $\lambda_1 \geq d_{avg}$ and the fact that if $\sum x_i^2 < y^2$, then $|\sum x_i^3| < |y^3|$.

Assume $e > n^2/4$. We know that $\lambda_1 \geq d_{avg} = 2e/n$, so

$$\lambda_1^2 \geq \frac{4e^2}{n^2} = e \cdot \frac{e}{n^2/4} > e.$$

We also know that $\sum \lambda_i^2 = 2e$, since it counts all closed walks of lengths 2, hence

$$\sum \lambda_i^2 = 2e < 2\lambda_1^2,$$

so

$$\sum_{i \geq 2} \lambda_i^2 < \lambda_1^2.$$

By the fact in the hint, this implies (using $\lambda_1 > 0$)

$$\left| \sum_{i \geq 2} \lambda_i^3 \right| < \lambda_1^3.$$

Therefore

$$6t = \sum \lambda_i^3 = \lambda_1^3 + \sum_{i \geq 2} \lambda_i^3 \geq \lambda_1^3 - \left| \sum_{i \geq 2} \lambda_i^3 \right| > 0.$$

4. (a) *The line graph $L(G)$ of a graph G has as vertices the edges of G, with an edge between two vertices of $L(G)$ if the corresponding edges in G touch at a vertex. For a connected k-regular graph G with*

$$Spec(G) = (k)^1 (\lambda)^{m(\lambda)} \cdots (\omega)^{m(\omega)},$$

show that

$$\text{Spec}(L(G)) = (2k-2)^1 (\lambda + k - 2)^{m(\lambda)} \cdots (\omega + k - 2)^{m(\omega)} (-2)^{|E(G)|-|V(G)|}.$$

Let B be the $n \times e$ incidence matrix B. We have (using that G is k-regular)

$$BB^T = A(G) + kI, \qquad B^T B = A(L(G)) + 2I,$$

On the other hand, the eigenvalues of any two such matrices are related. If $B^T Bv = \lambda v$, then

$$(BB^T)(Bv) = B(B^T Bv) = B(\lambda v) = \lambda(Bv),$$

so Bv is an eigenvector for BB^T with eigenvalue λ, unless $Bv = 0$ (which implies $\lambda = 0$). The same works the other way around. So BB^T and $B^T B$ have the same nonzero eigenvalues with the same multiplicities. The remaining eigenvalue is 0, with multiplicity whatever is left of the dimensions.
The eigenvalues of $A(G) + D$ are

$$\text{Spec}(BB^T) = (2k)^1 (\lambda + k)^{m(\lambda)} \cdots (\omega + k)^{m(\omega)},$$

so

$$\text{Spec}(B^T B) = (2k)^1 (\lambda + k)^{m(\lambda)} \cdots (\omega + k)^{m(\omega)} (0)^{e-n},$$

and

$$\text{Spec}(A(L(G))) = (2k-2)^1 (\lambda + k - 2)^{m(\lambda)} \cdots (\omega + k - 2)^{m(\omega)} (-2)^{e-n}.$$

Note that if the graph is bipartite, then $\omega = -k$, so $\omega + k - 2 = -2$ as well.

(b) *The complement $\overline{G}$ of G has the same vertices, but two vertices have an edge in $\overline{G}$ if and only if they do not have an edge in G.*
For a connected k-regular graph G with spectrum as in (a), show that

$$\mathrm{Spec}(\overline{G}) = (n-k-1)^1(-\lambda-1)^{m(\lambda)}\cdots(-\omega-1)^{m(\omega)}.$$

Because G is k-regular, it has $\lambda_1 = 1$ with multiplicity 1. Then clearly $\overline{G}$ is $n-k-1$-regular, so has largest eigenvalue $n-k-1$ (it could have higher multiplicity if one of the other eigenvalues ends up equalling this, corresponding to the fact that the complement graph may be disconnected).
Any other eigenvalue λ has an eigenvector v with $Jv = 0$, which gives

$$\overline{A}v = (J - I - A)v = Jv - Iv - Av = 0 - v - \lambda v = (-1-\lambda)v.$$

This tells us what the other eigenvalues of $\overline{G}$ are.

(c) *Give the spectrum of $\overline{L(K_n)}$. Write it out for $\overline{L(K_5)}$ and $\overline{L(K_8)}$ (the first should be familiar; the second will be used in the next lecture).*

$$\mathrm{Spec}(K_n) = (n-1)^1(-1)^{n-1}$$
$$\Rightarrow \mathrm{Spec}(L(K_n)) = (2n-4)^1(n-4)^{n-1}(-2)^{\binom{n}{2}-n}$$
$$\Rightarrow \mathrm{Spec}(\overline{L(K_n)}) = (\binom{n}{2} - 2n + 3)^1(3-n)^{n-1}(1)^{\binom{n}{2}-n}$$

$$\Rightarrow \mathrm{Spec}((\overline{L(K_5)}) = (3)^1(-2)^4(1)^5 = \mathrm{Spec}(\mathrm{Pet}), \quad \mathrm{Spec}((\overline{L(K_8)}) = (15)^1(-5)^7(1)^{20}$$

This does not imply that $\overline{L(K_5)} = \mathrm{Pet}$ (see question 2), but this is true. To see that, observe that $\overline{\mathrm{Pet}}$ has as vertices the 2-element subsets of [5], with an edge if they intersect. But this is exactly $L(K_5)$, since we can think of edges as 2-element subsets, which intersect if they touch at a vertex.