1. Show that if \( G \) has \( \geq 4 \) vertices and every two of its vertices have exactly one common neighbor, but there is no vertex adjacent to all the others, then \( G \) must be regular. 

(This is part of the proof of the windmill theorem.)

Please draw a picture when reading this solution. Note that the question is a bit weird, because this is part of a proof by contradiction, so a graph satisfying these condition does not actually exist. Still, I think it is a nice exercise.

We write \( f(u,v) \) for the common neighbor of two vertices. Take two non-adjacent vertices \( u, v \) and their unique common neighbor \( w = f(u,v) \). Then because \( uv \not\in E \), \( f(u,w) \) is not \( v \), and \( f(w,v) \) is not \( u \). Also \( f(u,w) \neq f(w,v) \) because that would give a \( C_4 \). So \( uwf(u,w) \) and \( wvf(w,v) \) form two triangles sharing only the vertex \( w \).

For any other neighbor \( t \) of \( u \), \( f(t,v) \) is not one of the 5 previous vertices, because in each case we would get a \( C_4 \). So we can pair up \( t \) with a neighbor of \( v \). We can repeat this for any other \( t' \in N(u) \), and then also \( f(t',v) \neq f(t,v) \) and \( f(t',v) \neq t \), because that would give a \( C_4 \). Repeating this we can pair off the remaining neighbors of \( u \) and \( v \), hence \( d(u) - 2 = d(v) - 2 \), showing that \( d(u) = d(v) \). So any 2 non-adjacent vertices have the same degree. But any vertex other than \( w \) is non-adjacent to at least one of \( u, v \), so these all have the same degree \( k \). And by the assumption at the start, \( w \) is not a neighbor of all vertices, so it will also have degree \( k \).

2. Find two non-isomorphic graphs that have the same spectrum, with one connected and the other one not connected. Conclude that connectedness cannot always be determined from the spectrum.

The graphs below both have spectrum \( (2)^1(0)^3(-2)^1 \).

3. (a) Show that if \( t \) is the number of triangles \( K_3 \) in a graph, then

\[
6t = \sum_{i=1}^{n} \lambda_i^3.
\]

By a lemma in the notes, this sum equals the number of closed walks of length 3. This counts every triangle 6 times, once for each starting point and direction of the walk.
(b) **Use this to (re)prove that if a graph has** \( > \frac{n^2}{4} \) **edges, then it contains a** \( K_3 \). **Hint:** Use \( \lambda_1 \geq d_{\text{avg}} \) and the fact that if \( \sum x_i^2 < y^2 \), then \( |\sum x_i^3| < |y^3| \).

Assume \( e > \frac{n^2}{4} \). We know that \( \lambda_1 \geq d_{\text{avg}} = \frac{2e}{n} \), so

\[
\lambda_1^2 \geq \frac{4e^2}{n^2} = \frac{e}{\frac{n^2}{4}} > e.
\]

We also know that \( \sum \lambda_i^2 = 2e \), since it counts all closed walks of lengths 2, hence

\[
\sum \lambda_i^2 = 2e < 2\lambda_1^2,
\]

so

\[
\sum_{i \geq 2} \lambda_i^2 < \lambda_1^2.
\]

By the fact in the hint, this implies (using \( \lambda_1 > 0 \))

\[
\left| \sum_{i \geq 2} \lambda_i^3 \right| < \lambda_1^3.
\]

Therefore

\[
6t = \sum \lambda_i^3 = \lambda_1^3 + \sum_{i \geq 2} \lambda_i^3 \geq \lambda_1^3 - \left| \sum_{i \geq 2} \lambda_i^3 \right| > 0.
\]

4. (a) **The line graph** \( L(G) \) **of a graph** \( G \) **has as vertices the edges of** \( G \), **with an edge between two vertices of** \( L(G) \) **if the corresponding edges in** \( G \) **touch at a vertex.**

For a connected \( k \)-regular graph \( G \) with

\[
\text{Spec}(G) = (k)^1(\lambda)^{m(\lambda)} \cdots (\omega)^{m(\omega)},
\]

show that

\[
\text{Spec}(L(G)) = (2k-2)^1(\lambda + k - 2)^{m(\lambda)} \cdots (\omega + k - 2)^{m(\omega)}(-2)^{|E(G)| - |V(G)|}.
\]

Let \( B \) be the \( n \times e \) incidence matrix \( B \). We have (using that \( G \) is \( k \)-regular)

\[
BB^T = A(G) + kI, \quad B^TB = A(L(G)) + 2I,
\]

On the other hand, the eigenvalues of any two such matrices are related. If \( B^TBv = \lambda v \), then

\[
(BB^T)(Bv) = B(B^TBv) = B(\lambda v) = \lambda (Bv),
\]

so \( Bv \) is an eigenvector for \( BB^T \) with eigenvalue \( \lambda \), unless \( Bv = 0 \) (which implies \( \lambda = 0 \)). The same works the other way around. So \( BB^T \) and \( B^TB \) have the same nonzero eigenvalues with the same multiplicities. The remaining eigenvalue is 0, with multiplicity whatever is left of the dimensions.

The eigenvalues of \( A(G) + D \) are

\[
\text{Spec}(BB^T) = (2k)^1(\lambda + k)^{m(\lambda)} \cdots (\omega + k)^{m(\omega)},
\]

so

\[
\text{Spec}(B^TB) = (2k)^1(\lambda + k)^{m(\lambda)} \cdots (\omega + k)^{m(\omega)}(0)^{e-n},
\]

and

\[
\text{Spec}(A(L(G))) = (2k-2)^1(\lambda + k - 2)^{m(\lambda)} \cdots (\omega + k - 2)^{m(\omega)}(-2)^{e-n}.
\]

Note that if the graph is bipartite, then \( \omega = -k \), so \( \omega + k - 2 = -2 \) as well.
(b) The complement $\overline{G}$ of $G$ has the same vertices, but two vertices have an edge in $\overline{G}$ if and only if they do not have an edge in $G$.

For a connected $k$-regular graph $G$ with spectrum as in (a), show that

$$\text{Spec}(\overline{G}) = (n - k - 1)^1(-\lambda - 1)^m(\lambda) \cdots (-\omega - 1)^m(\omega).$$

Because $G$ is $k$-regular, it has $\lambda_1 = 1$ with multiplicity 1. Then clearly $\overline{G}$ is $n-k-1$-regular, so has largest eigenvalue $n-k-1$ (it could have higher multiplicity if one of the other eigenvalues ends up equalling this, corresponding to the fact that the complement graph may be disconnected).

Any other eigenvalue $\lambda$ has an eigenvector $v$ with $Jv = 0$, which gives

$$Av = (J - I - A)v = Jv - Iv - Av = 0 - v - \lambda v = (-1 - \lambda)v.$$

This tells us what the other eigenvalues of $\overline{G}$ are.

(c) Give the spectrum of $L(K_n)$. Write it out for $L(K_5)$ and $L(K_8)$ (the first should be familiar; the second will be used in the next lecture).

$$\text{Spec}(K_n) = (n - 1)^1(-1)^{n-1}$$

$\Rightarrow$ Spec($L(K_n)$) = $(2n - 4)^1(n - 4)^{n-1}(2)^{-n}$

$\Rightarrow$ Spec($L(K_n)$) = $(\binom{n}{2} - 2n + 3)^1(3 - n)^{n-1}(1)^{-n}$

$\Rightarrow$ Spec($L(K_5)$) = $(3)^1(-2)^4(1)^5 = \text{Spec(Pet)}$, Spec($L(K_8)$) = $(15)^1(-5)^7(1)^{20}$

This does not imply that $L(K_5) = \text{Pet}$ (see question 2), but this is true. To see that, observe that Pet has as vertices the 2-element subsets of [5], with an edge if they intersect. But this is exactly $L(K_5)$, since we can think of edges as 2-element subsets, which intersect if they touch at a vertex.