ADM 2013 – Problem Set 8 – Bonus Solutions

*5. *Prove that the Petersen graph does not have a Hamilton cycle.*

Some of you handed in solutions with case-by-case proofs, which is not wrong, but the nicest proof uses, wait for it, *eigenvalues!*
Because the Petersen graph is 3-regular, if it had a Hamilton cycle then it would decompose into a 2-regular graph C (the cycle) and a 1-regular graph M (the remaining edges, which must form a matching). For the matrices this would mean

$$A(\text{Pet}) = A(M) + A(C).$$

As in the proof in the lecture about decomposing K_{10} into 3 Petersens, we can get a contradiction from this equation by looking for common eigenvectors for the different matrices.
First recall that

$$\text{Spec}(\text{Pet}) = (3)^1(1)^5(-2)^4 \quad \text{and} \quad \text{Spec}(M) = (1)^5(-1)^5;$$

the second follows from the fact that it is a disjoint union of 5 copies of K_2.
Both have the all-ones vector j as an eigenvector for the largest eigenvalue. That leaves 9 dimensions for the other eigenvectors (in an orthogonal basis). But the eigenvalue 1 of Pet and the eigenvalue -1 of M both have multiplicity 5, which means they must have a common eigenvector v, which is orthogonal to j. Then

$$A_C v = A_{\text{Pet}} v - A_M v = 1 \cdot v - (-1) \cdot v = 2v.$$

So v is an eigenvector for the eigenvalue 2 of C. That is indeed an eigenvalue for C, but we already have the eigenvector j for it, so $m_C(2) \geq 2$. However, as we saw in the last problem set, if a regular graph is connected, then its largest eigenvalue has multiplicity 1. So C is not connected, hence is not a Hamilton cycle.
Note that because the Petersen graph contains no C_3 or C_4, we have even proved that its only 2-regular subgraphs are C_5s.

*6. *Reprove the windmill theorem without using linear algebra, using the following outline. We've already proved without linear algebra that G is k-regular and $n = k^2 - k + 1$. Let $f(m)$ be the number of m-walks from a fixed vertex to itself. Then*

$$f(m) = (k-1)f(m-2) + k^{m-2}.$$

Pick a prime p dividing $k-1$. Then it follows from the formula for f that the total number of closed p-walks is $\equiv 1$ mod p, which gives a contradiction.

First we prove the formula for $f(m)$. Starting from the fixed vertex v, pick a random sequence of $m-2$ adjacent edges; this can be done in k^{m-2} ways. If we end up at a vertex other than v, then there is a unique path of 2 edges that we can use to finish the m-path, via the unique common neighbor. If we end up back at v after the $m-2$ edges, we go back and forth to a neighbor to get an m-path. If we fix some favorite neighbor u of v to go back and forth to, then this gives exactly k^{m-2} m-walks, but we have missed out on the $(m-2)$-walks that go from v to itself and then back and forth to a neighbor other than u. There are exactly $(k-1)f(m-2)$ such $(m-2)$-walks. We pick a prime p dividing $k-1$. Then $k-1 \equiv 0 \bmod p$, so

$$f(p) \equiv k^{p-2} \equiv 1 \bmod p.$$

Then the total number of closed p-walks (from some vertex to itself) is

$$\sum_{v \in V(G)} f(p) = n \cdot f(p) = (k(k-1)+1)f(p) \equiv (0+1) \cdot 1 = 1 \bmod p.$$

But this is a contradiction. The "same" p-walk will be counted $2p$ times: Once for each choice of starting point and direction. Therefore the total number of p-walks, counted in this way, will be divisible by $2p$, so will be $\equiv 0 \bmod p$.