1. Prove the following facts about eigenvalues of graphs.
   
   (a) If $G$ has at least one edge, then it has a negative eigenvalue.
   
   (b) Suppose $G$ is connected. If an eigenvector of $G$ is real and nonnegative (each entry is $\geq 0$), then it is positive (each entry is $> 0$).
   
   (c) If an eigenvalue of a graph is in $\mathbb{Q}$, then it is in $\mathbb{Z}$.
   
   **Hint:** The characteristic polynomial $\det(A - xI)$ has integer coefficients.

2. Determine the spectrum of $P_n$, the path with $n$ vertices and $n - 1$ edges.
   
   **Hint:** If an eigenvector of a cycle has value 0 at a vertex, then you can remove that vertex. None of the eigenvectors of the cycle from class have a value 0, but if the cycle is even, then there are eigenvalues with multiplicity 2. So any linear combination of two eigenvectors for such an eigenvalue will be an eigenvector; some of these will have a value 0.

3. Prove that if $G$ is $d$-regular, then the multiplicity of the largest eigenvalue $\lambda_1$ equals the number of connected components of $G$.
   
   **Hint:** Use the equality $\lambda x_u = \sum_{v \in N(u)} x_v$ for eigenvectors $(x_v)_{v \in V(G)}$.

4. Prove that if $G$ is connected, then the diameter of $G$ is strictly less than its number of distinct eigenvalues.

*5. (a) The line graph $L(G)$ of a graph $G$ has as vertices the edges of $G$, with an edge between two vertices of $L(G)$ if the corresponding edges in $G$ touch at a vertex. For a regular graph $G$, give the spectrum of $L(G)$ in terms of the spectrum of $G$.
   
   **Hint:** Let $B$ be the incidence matrix of $A$. Then $BB^T$ and $B^TB$ have mostly the same eigenvalues, and they have something to do with $A_G$ and $A_{L(G)}$.

   (b) The complement $\overline{G}$ of $G$ has the same vertices, but two vertices have an edge in $\overline{G}$ if and only if they do not have an edge in $G$.
   
   For a regular graph $G$, give the spectrum of $\overline{G}$ in terms of the spectrum of $G$.

   (c) Use (a) and (b) (and nothing else!) to find the spectrum of the Petersen graph.

*6. Find the spectrum of the cube graph $Q_n$, which has vertices all subsets of $[n]$, with $S$ and $T$ connected by an edge if $|S \cap T| = 1$. (Equivalently, the vertices are $n$-dimensional 01-vectors, with two of them connected if they differ in one entry.)