1. **Prove using induction that a planar graph with \( n \geq 3 \) vertices has at most \( 3n - 6 \) edges.**

   First we show the following claim: Every planar graph contains an edge \( uv \) such that \( u \) and \( v \) have at most two common neighbors. Take any edge \( uv \), and suppose that \( u \) and \( v \) have more than two common neighbors. Then we can find common neighbors \( w \) and \( x \) of \( u \) and \( v \) such that \( x \) lies inside the triangle \( uvw \). We repeat the same for the edge \( ux \): if \( u \) and \( x \) have more than two common neighbors, then we can find two of them such that one lies inside the triangle formed by \( ux \) and the other. Moreover, this second triangle must be contained in the triangle \( uvw \). Repeating this procedure, we obtain smaller and smaller triangles, and eventually we must find an edge connecting two vertices with at most two common neighbors.

   To prove that a planar graph \( G \) with \( n \geq 3 \) vertices has at most \( 3n - 6 \) edges, we proceed by induction on \( n \). For \( n = 3 \), the statement is obvious. For \( n \geq 4 \), we use the claim to find an edge \( uv \) such that \( u \) and \( v \) have at most two common neighbors. Then we remove the vertices \( u \) and \( v \) and replace them by a new vertex \( v' \) connected to those vertices of \( G - \{ u, v \} \) that were connected to any of \( u, v \). The graph \( G' \) thus obtained is smaller by one vertex and at most three edges, and so the induction hypothesis applied to \( G' \) proves the statement for \( G \).

2. **Show that the bound on \( \text{cr}(G) \) in the crossing lemma is optimal up to a constant factor.**

   The crossing number of a complete graph with \( k \) vertices is at most \( 3(k) < 1/8 k(k-1)^3 \), as each subset of 4 vertices generates at most 3 crossings. Let the graph \( G \) consist of \( n \) disjoint complete graphs of size \( k \). Clearly, its crossing number is equal to the sum of the crossing numbers of these complete graphs, and thus \( \text{cr}(G) < \frac{n}{8} \cdot 1/8 k(k-1)^3 = \frac{1}{64} n(k-1)^3 \).

   The number of edges in \( G \) is \( m = \frac{n}{k} \cdot \frac{k}{2} = \frac{1}{2} n(k-1) \). Therefore, we have \( \text{cr}(G) < m^3/n^2 \).

3. **Show that the bound on the number of point-line incidences in Szemerédi-Trotter theorem is optimal up to a constant factor.**

   To this end, for \( r, s \in \mathbb{N} \), find a configuration of \( 2r^2 s \) points and \( rs^2 \) lines in the plane such that the number of incidences between these points and lines is equal to \( r^2 s^2 \).

   Let \( P = \{1, \ldots, r\} \times \{1, \ldots, 2rs\} \) be the set of points, and \( L = \{\ell(a, b): a \in \{1, \ldots, s\} \text{ and } b \in \{1, \ldots, rs\}\} \) be the set of lines, where \( \ell(a, b) \) is the line given by equation \( y = ax + b \). Clearly, we have \( |P| = 2r^2 s \), \( |L| = rs^2 \), and each line in \( L \) passes through exactly \( r \) points from \( P \). Therefore, the total number of point-line incidences is \( r^2 s^2 = 2^{-2/3} |P|^{2/3} |L|^{2/3} \).

4. **Prove that there is an absolute constant \( c > 0 \) with the following property: for any set \( X \) of \( n \) points in the plane, the number of lines passing through at least \( k \geq 2 \) points in \( X \) is at most \( c(n^2/k^3 + n/k) \).**

   Let \( P \) be a set of \( n \) points in the plane and \( L \) be a set of those lines that pass through at least \( k \) points from \( P \). It is clear that \( |L| \leq n^2 \). Therefore, if \( k \leq 4 \), then the statement holds for \( c = 1/64 \). Thus suppose \( k \geq 5 \). Let \( m = |L| \). Suppose that \( m > c(n^2/k^3 + n/k) \) for some constant \( c > 0 \) to be chosen later. This implies that \( m^{1/3} > c^{1/3} n^{2/3}/k \) and \( m > cn/k \). The number of incidences between points in \( P \) and lines in \( L \) is

\[
|I| \geq km = 4m + \frac{k - 4}{2} m + \frac{k - 4}{2} m > 4m + c^{1/3} \frac{k - 4}{2k} m^{2/3} n^{2/3} + \frac{k - 4}{2k} n
\]

\[
\geq 4m + \frac{c^{1/3}}{10} m^{2/3} n^{2/3} + \frac{c}{10} n.
\]
On the other hand, by Szemerédi-Trotter theorem, we have $|I| \leq 4(m^{2/3}n^{2/3} + m + n)$. The two inequalities give contradiction for $c = 40^3$.

5. In a manner similar to the proof of Szemerédi-Trotter theorem, prove that there is an absolute constant $c > 0$ such that the number of incidences between $n$ points and $m$ unit circles in the plane is at most $c(m^{2/3}n^{2/3}+m+n)$. Be careful in handling possible multiple edges in the graph considered in the proof.

Let $P$ be a set of $n$ points and $C$ be a set of $m$ unit circles in the plane. Consider the following multigraph $G$ (a graph with possible multiple edges) drawn in the plane. The vertices of $G$ are the points in $P$. Each circle in $C$ passing through $k$ points from $P$ is split by these points into $k$ arcs. Let $k - 1$ of these arcs be edges of $G$ (that is, we disregard one arc). Clearly, the total number of point-circle incidences is $|I| = |E(G)| + m$. In order to apply the crossing lemma as in the proof of Szemerédi-Trotter theorem, we first need to get rid of multiple edges. Note that any two vertices are connected by at most two edges, because there are at most two unit circles passing through these points, each giving rise to at most one edge. Whenever two vertices are connected by two edges, keep only one of them and remove the other. This way a new graph $G'$ with $|E(G')| \geq \frac{1}{2}|E(G)| \geq \frac{1}{2}(|I| - m)$ and without multiple edges is obtained. If $|E(G')| < 4|V(G')| = 4n$, then we have $|I| \leq 8n + m$. Otherwise, we use the crossing lemma for $G'$ and obtain

$$\text{cr}(G') \geq \frac{|E(G')|^3}{64|V(G')|^2} \geq \frac{(|I| - m)^3}{512n^2}.$$  

On the other hand, any two circles can cross at most twice, so we have $\text{cr}(G') \leq 2\binom{m}{2} < m^2$. Composing the two inequalities, we obtain $|I| \leq 8m^{2/3}n^{2/3} + m$. Therefore, in both cases we have an upper bound of $8(m^{2/3}n^{2/3} + m + n)$ on the number of incidences.

6. Prove that there is an absolute constant $c > 0$ with the following property: for any set $X$ of $n$ points in the plane, the number of pairs of points in $X$ at distance exactly 1 is at most $cn^{4/3}$.

For each point from $X$, draw this point and the circle centered at this point. This way a configuration of points and unit circles in the plane is obtained, for which we can apply the result of problem 5. As a consequence, the number of point-circle incidences is at most $8(n^{4/3} + 2n) \leq 24n^{4/3}$. Since each pair of points at distance 1 generates two such incidences, the number of pairs of points at distance 1 is at most $12n^{4/3}$.  

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