Advanced Discrete Mathematics 2013 – Problem Set 4

Notation: $[n] = \{1, 2, \ldots, n\}$, $[n]^{(k)} = \{S \subset [n] : |S| = k\}$, $\mathcal{P}(X) = 2^X = \{S : S \subset X\}$.

1. Suppose that for $i \in [m]$ we have polynomials $f_i \in \mathbb{F}[x_1, \ldots, x_n]$ and vectors v_i such that
$$f_i(v_j) = 0 \quad \text{for all } j > i,$$
$$f_i(v_i) \neq 0 \quad \text{for all } i.$$
Prove that the f_i are linearly independent.

2. Give tight examples for Oddtown and Eventown, and for Problem 6 and Problem 7.

3. A set system $(X, \mathcal{S})$ is called *intersecting* if $|S \cap T| \geq 1$ for all $S, T \in \mathcal{S}$.
Show that the maximum size of an intersecting set system on X is $2^{|X|-1}$.
(This means you have to show that this is an upper bound, and give a tight example.)

4. Suppose $A \subset \mathbb{F}_3^n$ has the property that for all distinct vectors $a, b \in A$, there is a coordinate $i \in [n]$ such that $a_i - b_i \equiv 1 \bmod 3$. Show that $|A| \leq 2^n$.
Hint: Construct linearly independent polynomials.

5. Show that the graph G defined by
$$V(G) = [m]^{(3)}, \qquad E(G) = \{ST : |S \cap T| = 1\}$$
is a k-Ramsey graph, ie G contains no K_k and no k independent vertices.

6. * **Reverse Oddtown**
Let $|X| = n$. Suppose $(X, \mathcal{S})$ is a set system such that $|S \cap T|$ is odd for all $S \neq T \in \mathcal{S}$, and $|S|$ is even for all $S \in \mathcal{S}$.
Prove that $|\mathcal{S}| \leq n$ if n is odd, and $|\mathcal{S}| \leq n - 1$ if n is even.

7. * **Uniform L-intersecting set systems**
Let $|X| = n$ and $L \subset [0, n-1]$ with $|L| = l$. Suppose $(X, \mathcal{S})$ is a *uniform* L-intersecting set system, ie all $S \in \mathcal{S}$ have the same size k, and $|S \cap T| \in L$ for all $S \neq T \in \mathcal{S}$.
Prove that $|\mathcal{S}| \leq \binom{n}{l}$.
Hint: Do the same as for the nonuniform case from class, but also use the polynomials
$$g_I(x) = \left(\sum_{i=1}^{n} x_j - k\right) \cdot \prod_{i \in I} x_i, \qquad I \subset [n],\ |I| < l.$$

8. * **Turn on the lights!**
For a vertex v of a graph G, let $\widetilde{N}(v) = \{v\} \cup \{u \in V(G) : uv \in E(G)\}$.
Show that there exists $S \subset V(G)$ such that $|S \cap \widetilde{N}(v)|$ is odd for all $v \in V(G)$.
To put it more vividly, imagine that at each v there is a lightbulb and a light switch that can change the lights of $\widetilde{N}(V)$ (from off to on or on to off). Then if all the lights are off, we can pick a subset of switches that turns on all the lights at the same time.