Advanced Discrete Mathematics 2013 – Problem Set 4 – Bonus Solutions

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Notation: $[n] = \{1, 2, \ldots, n\}$, $[n]^{(k)} = \{S \subset [n] : |S| = k\}$, $\mathcal{P}(X) = 2^X = \{S : S \subset X\}$.

6. * **Reverse Oddtown**
Let $|X| = n$. Suppose $(X, \mathcal{S})$ is a set system such that $|S \cap T|$ is odd for all $S \neq T \in \mathcal{S}$, and $|S|$ is even for all $S \in \mathcal{S}$.
Prove that $|\mathcal{S}| \leq n$ if n is odd, and $|\mathcal{S}| \leq n - 1$ if n is even.

I will give 2 proofs. The first was found by Antoine Imboden, the second by Kokollari Kreshnik.
The only way of solving this question (as far as I know) is by combining several little arguments, and the result is a bit long and not so elegant. Each trick by itself is not that hard to come up with, but it can be a bit of a puzzle to combine these into a correct logical structure, together with the usual arguments about linear independence or matrix ranks. Let me list these tricks:

- Adding an element to each $S \in \mathcal{S}$ will give an Oddtown example (but note that removing an element won't do the same, because there need not be an element contained in all S).
- Taking the complements of all S will give an Oddtown example if n is odd, and another Reverse Oddtown example if n is even.
- Taking the sum mod 2 of all the v_S gives a new vector u, which corresponds to a new set. If $|\mathcal{S}|$ is even, then adding this set to $\mathcal{S}$ will give a new Reverse Oddtown example.

Proof 1: If n is odd, then we can deduce this from the Oddtown theorem. Given $\mathcal{S}$ as above, define $\overline{\mathcal{S}} = \{\overline{S} : S \in \mathcal{S}\}$. Because n is odd and $|S|$ is even, $|\overline{S}|$ will be odd. For the intersection we have (mod 2)

$$|\overline{S} \cap \overline{T}| = n - |S \cup T| = n - |S| - |T| + |S \cap T| \equiv 1 - 1 - 1 + 1 \equiv 0.$$

So by the Oddtown theorem $|\mathcal{S}| = |\overline{\mathcal{S}}| \leq n$.

Now assume n is even. We will show that the usual vectors v_S are linearly independent (this would actually also work for n odd, but the argument above was nicer).
Suppose $\sum \lambda_s v_s = 0$. Then

$$0 = \langle v_T, \sum \lambda_S v_S \rangle = \sum_{S \neq T} \lambda_S \langle v_S, v_T \rangle.$$

If we subtract two such equations for $T \neq T'$, we get $\lambda_T = \lambda_{T'}$. So in fact we have $\lambda_T = \lambda$ for all $T \in \mathcal{S}$, which means that either all $\lambda_T = 0$ and we have proved linear independence, or else

$$\sum_{S \in \mathcal{S}} v_S = 0.$$

It follows that

$$0 = \langle v_T, \sum v_S \rangle = \sum \langle v_T, v_S \rangle = \sum_{S \neq T} 1 = |\mathcal{S}| - 1,$$

which means that $|\mathcal{S}|$ is odd.
On the other hand, if we consider $\overline{\mathcal{S}}$ again, you can check that we now get another Reverse Oddtown set system. Hence by the same argument as above we also have

$$0 = \sum_{\overline{S} \in \overline{\mathcal{S}}} v_{\overline{S}} = \sum_{S \in \mathcal{S}} v_{\overline{S}}.$$

Observe that $v_S + v_{\overline{S}} = v_X$, the vector with all entries 1, so we get

$$0 = \sum_{S \in \mathcal{S}} v_S + \sum_{S \in \mathcal{S}} v_{\overline{S}} = \sum_{S \in \mathcal{S}} (v_S + v_{\overline{S}}) = |\mathcal{S}| \cdot v_X,$$

which implies that $|\mathcal{S}|$ is even, a contradiction. We have derived a contradiction from $\sum v_s = 0$, so we must have all $\lambda_S = 0$, proving that the v_S are linearly independent. Since they lie in a vector space of dimension n, it follows that there at most n of them.

This proves that $|\mathcal{S}| \leq n$, but we want to show that $|\mathcal{S}| \leq n - 1$ for n even. So suppose that $|\mathcal{S}| = n$ and is even.
Consider the vector

$$u := \sum_{S \in \mathcal{S}} v_S \pmod 2$$

that we saw above. It corresponds to the set S_u that contains x if and only if x is in an odd number of sets from $\mathcal{S}$ (but this definition would be hard to come up with if we weren't using vector notation). We will show that we can add S_u to $\mathcal{S}$ and still have a Reverse Oddtown set system. The vector u satisfies

$$\langle u, v_T \rangle = \sum_{S \in \mathcal{S}} \langle v_S, v_T \rangle = \sum_{S \neq T} 1 = |\mathcal{S}| - 1 \equiv 1$$

by the assumption that $|\mathcal{S}|$ is even. This implies that S_u is distinct from all $T \in \mathcal{S}$ and has odd intersection with all of them. Also

$$\langle u, u \rangle = \sum_{(S,T) \in \mathcal{S} \times \mathcal{S}} \langle v_S, v_T \rangle = \sum_{S \neq T \in \mathcal{S}} 2 \langle v_S, v_T \rangle \equiv 0.$$

This shows that $|S_u|$ is even. Therefore $\mathcal{S} \cup \{S_u\}$ is also a Reverse Oddtown set system. But it would have size $n+1$, contradicting what we proved before. So our assumption $|\mathcal{S}| = n$ must be false, and we can finally conclude that $|\mathcal{S}| \leq n - 1$.

Proof 2: If n is odd, we do the same as above, so let's assume n is even.
We can prove easily that $|\mathcal{S}| \leq n+1$, by adding a new element to each $S \in \mathcal{S}$, which gives an Oddtown example with $|X| = n+1$. So we will be done if we can show that $|\mathcal{S}| = n$ or $n+1$ lead to contradictions.
We look at the usual $n \times |\mathcal{S}|$ incidence matrix A which has the columns v_S, and the $|\mathcal{S}| \times |\mathcal{S}|$ adjacency matrix (over $\mathbb{F}_2$)

$$B = A^T A \equiv J - I \bmod 2.$$

We make the following observations (over $\mathbb{F}_2$):

- *rk*$(A) \leq n - 1$*:* The sum of the row vectors of A equals 0, because its entries are exactly the sizes of the $S \in \mathcal{S}$, which are even. So the row vectors are linearly dependent, and A cannot have full rank.

- *If $|\mathcal{S}|$ is even, then rk$(B) = |\mathcal{S}|$:* The span of the column vectors of $J - I$ contains each standard basis vector e_i. Indeed, adding up all the column vectors and dividing by $|\mathcal{S}| - 1$ gives the vector with all entries 1, then subtracting the i-th column vectors leaves us with e_i.
- *If $|\mathcal{S}|$ is odd, then rk$(B) = |\mathcal{S}| - 1$:* Again the sum of the column vectors of $J - I$ equals $|\mathcal{S}| - 1$ times the vectors with all entries 1, but now this equals the zero vector because $|\mathcal{S}| - 1$ is even. So $\mathrm{rk}(B) < |\mathcal{S}|$. But it has a submatrix of size $|\mathcal{S}| - 1 \times |\mathcal{S}| - 1$ which is of the form $J - I$, so by the argument in the previous observation, this submatrix has full rank, so $\mathrm{rk}(B) = |\mathcal{S}| - 1$.

Using that n is even, this lets us derive a contradiction from $|\mathcal{S}| = n$ or $n+1$ as follows:

- *If $|\mathcal{S}| = n + 1$:* Then $|\mathcal{S}|$ is odd, so by the third and first observations above we have
$$|\mathcal{S}| = \mathrm{rk}(B) + 1 \leq \mathrm{rk}(A) + 1 \leq n - 1 + 1 = n = |\mathcal{S}| - 1,$$
a contradiction.
- *If $|\mathcal{S}| = n$:* Then $|\mathcal{S}|$ is even, so by the second and first observations we have
$$|\mathcal{S}| = \mathrm{rk}(B) \leq \mathrm{rk}(A) \leq n - 1 = |\mathcal{S}| - 1,$$
also a contradiction.

This completes the proof that $|\mathcal{S}| \leq n - 1$.

7. * **Uniform L-intersecting set systems**
Let $|X| = n$ and $L \subset [n]$ with $|L| = l$. Suppose $(X, \mathcal{S})$ is a uniform L-intersecting set system, ie all $S \in \mathcal{S}$ have the same size k, and $|S \cap T| \in L$ for all $S \neq T \in \mathcal{S}$.
Prove that $|\mathcal{S}| \leq \binom{n}{l}$.

We use the same polynomials

$$f_S(\bar{x}) = \prod_{\substack{k \in L: \\ k < |S|}} (\langle \bar{x}, v_S \rangle - k)$$

from the proof of the nonuniform theorem that we saw in class. As we saw there, their purifications $\widetilde{f_S}$ are linearly independent and contained in the subspace

$$U = \text{span}\left(\left\{\prod_{i \in I} x_i : I \subset [n],\ |I| \leq l\right\}\right),$$

which has dimension $\sum_{i=0}^{l} \binom{n}{i}$.
To get the bound that we want, we will show that the $\sum_{i=0}^{l-1} \binom{n}{i}$ polynomials

$$g_I(x) = \left(\sum_{i=1}^{n} x_j - k\right) \cdot \prod_{i \in I} x_i, \qquad I \subset X = [n],\ |I| < l,$$

from the hint are also in U, and that $\{\widetilde{f_S}, g_I\}$ is linearly independent. Then it will follow that

$$|\mathcal{S}| = \#\widetilde{f_S} \leq \dim(U) - \#g_I = \sum_{i=0}^{l} \binom{n}{i} - \sum_{i=0}^{l-1} \binom{n}{i} = \binom{n}{l}.$$

We make the following observations:

- $g_I(v_S) = 0$ for all $S \in \mathcal{S}$, because then $k = |S| = \sum (v_S)_i$, so the first factor of g_I equals zero.

- $g_I(v_I) \neq 0$ for all $I \subset X$, $|I| < l$.

- $g_I(v_J) = 0$ for any $J \neq I \subset X$ with $|J| \leq |I|$, because there must be some $i \in I \backslash J$, so this factor x_i will make g_I zero.

Now order $\{I \subset X : |I| < l\} \cup \mathcal{S}$ as follows: Order $\{I \subset X : |I| < l\}$ by decreasing size, so $I < J$ if $|I| > |J|$ (and arbitrarily ordered when $|I| = |J|$), then put the $S \in \mathcal{S}$ at the end in an arbitrary order.
Set $h_I = g_I$ if $|I| < l$, and $h_I = \widetilde{f_I}$ for $I \in \mathcal{S}$. Then the polynomials h_I satisfy the conditions of problem 1: $h_I(v_I) \neq 0$ and

$$I < J \quad \Rightarrow \quad h_I(v_J) = 0.$$

So they are linearly independent, which is what we wanted, since $\{h_I\} = \{g_I, f_S\}$.
(I used this weird order to squeeze it into the form of problem 1, it might be easier to prove independence more directly.)

8. * **Turn on the lights!**
For a vertex v of a graph G, let $\widetilde{N}(v) = \{v\} \cup \{u \in V(G) : uv \in E(G)\}$.
Show that there exists $S \subset V(G)$ such that $|S \cap \widetilde{N}(v)|$ is odd for all $v \in V(G)$.

To put it more vividly, imagine that at each v there is a lightbulb and a light switch that can change the lights of $\widetilde{N}(V)$ (from off to on or on to off). Then if all the lights are off, we can pick a subset of switches that turns on all the lights at the same time.

We will do everything over $\mathbb{F}_2$. Let A be the adjacency matrix of G, let y be the vector with $|V(G)|$ 1s, and for a subset $S \subset V(G)$ write v_S for its characteristic vector.
A set $S \subset V(G)$ is as required if (mod 2)

$$(A + I)v_S = y.$$

Indeed, this says that if for each vertex we count the number of elements of S that are in $\widetilde{N}(S)$, we get 1 mod 2.
To look at it differently, this says that $y \in \text{im}(A + I)$. We claim that $\text{im}(A + I)$ is orthogonal to $\ker(I + A)$. Indeed, suppose $(A + I)z = y$ and $(A + I)x = 0$, then

$$\langle y, x\rangle = \langle (A + I)z, x\rangle = \langle z, (A + I)^T x\rangle = \langle z, (A + I)x\rangle = 0.$$

So to prove that an S as above exists it suffices to prove that y is orthogonal to $\ker(I + A)$.
Let $u \in \ker(I + A)$, so $(A + I)u = 0$. Since u is a 0/1-vector, it defines a subgraph H of G, taking all vertices for which the corresponding entry of u is 1, and taking all edges between two of those vertices. Because $(A + I)u = 0$, we have for each $x \in V(H)$ that

$$0 = \sum_{y \in V(H)} (A + I)_{xy} = 1 + \sum_{y \in V(H)} A_{xy} = 1 + \deg_H(x).$$

This means that $\deg_H(x)$ is odd for each $x \in V(H)$. But from graph theory we know that

$$\sum_{x \in V(H)} \deg_H(x) = 2|E(H)|,$$

so $|V(H)|$ must be even. Therefore

$$\langle u, y\rangle = 0,$$

which means that y is orthogonal to u. This holds for each $u \in \ker(A + I)$, so y is orthogonal to $\ker(I + A)$.