Advanced Discrete Mathematics 2013 – Problem Set 4 – Solutions

Notation: $[n] = \{1, 2, \ldots, n\}$, $[n]^{(k)} = \{S \subset [n] : |S| = k\}$, $\mathcal{P}(X) = 2^X = \{S : S \subset X\}$.

1. *Suppose that for $i \in [m]$ we have polynomials $f_i \in \mathbb{F}[x_1, \ldots, x_n]$ and vectors v_i such that*
$$f_i(v_j) = 0 \quad \text{for all } j > i,$$
$$f_i(v_i) \neq 0 \quad \text{for all } i.$$
Prove that the f_i are linearly independent.

   Suppose $\sum \lambda_i f_i = 0$. Plugging in v_1 gives
$$0 = \sum \lambda_i f_i(v_1) = \lambda_1 f_1(v_1)$$
since $f_i(v_1) = 0$ for all $i > 1$. But $f_1(v_1) \neq 0$, so $\lambda_1 = 0$.
Next plugging in v_2 gives $\lambda_2 = 0$, using that λ_1 is already gone. Repeating this will give that all $\lambda_i = 0$.

2. *Give tight examples for Oddtown and Eventown, and for Reverse Oddtown (Problem 6) and uniform L-intersecting set systems (Problem 7).*

   - Oddtown: Just take $\mathcal{S} = \{\{1\}, \{2\}, \ldots, \{|X|\}\}$. Many more are possible.
- Eventown: Let $\mathcal{T} = \{\{1, 2\}, \{3, 4\}, \ldots, \{2 \cdot \lfloor \frac{n}{2} \rfloor - 1, 2 \cdot \lfloor \frac{n}{2} \rfloor\}$ and take all combinations of these 2-element sets, ie
$$\mathcal{S} = \left\{ \bigcup_{S \in T} S : T \subset \mathcal{T} \right\}.$$
- Reverse Oddtown: For even n, pick an $x \in X$ and take the sets $\{x, y\}$ for all $x \neq y \in X$. For odd n, do the same but add the subset $X \backslash \{x\}$.
- Uniform L-intersecting set systems: Take $\mathcal{S} = [n]^{(l)}$ and $L = \{0, 1, \ldots, l-1\}$.

3. *A set system $(X, \mathcal{S})$ is called intersecting if $|S \cap T| \geq 1$ for all $S, T \in \mathcal{S}$. Show that the maximum size of an intersecting set system on X is $2^{|X|-1}$.*

   For a tight example, pick an $x \in X$ and take the sets $\{x\} \cup S$ for all $S \subset \mathcal{P}(X \backslash \{x\})$.
For the bound, we can partition 2^X into pairs $S, \overline{S}$, where $\overline{S} = \{x \in X : x \notin S\}$ is the complement of S. An intersecting set system can only contain one of each pair $S, \overline{S}$, hence its size is no greater than the number of such pairs, which is $2^{|X|-1}$.

4. *Suppose $A \subset \mathbb{F}_3^n$ has the property that for all distinct vectors $a, b \in A$, there is a coordinate $i \in [n]$ such that $a_i - b_i \equiv 1$ mod 3. Show that $|A| \leq 2^n$.*

   We define the polynomials $f_a \in \mathbb{F}_3[\bar{x}] = \mathbb{F}_3[x_1, \ldots, x_n]$ for $a \in A$ by
$$f_a(\bar{x}) = \prod_{i=1}^{n} (a_i - x_i - 1).$$
Then we have for $a \neq b \in A$ that $f_a(b) \equiv 0$, exactly by the property of A in the question. We also have $f_a(a) = (-1)^n \not\equiv 0$. So by problem 1, the f_a are linearly independent in the infinite-dimensional vector space $\mathbb{F}_3[\bar{x}]$. That doesn't help, but fortunately they are contained in the finite-dimensional subspace
$$U = \text{span}\left(\left\{ \prod_{i \in I} x_i : I \subset [n] \right\} \right),$$

since every f_a is a linear combination of such pure monomials (or in other words f_a is multilinear). So we get

$$|A| = \# f_a \leq \dim(U) = \#\{I \subset [n]\} = 2^n.$$

Note that here we don't need the "purification" trick that we needed for the theorems about L-intersecting sets.

5. *Show that the graph G defined by*

$$V(G) = [m]^{(3)}, \qquad E(G) = \{ST : |S \cap T| = 1\}$$

is a k-Ramsey graph, ie G contains no K_k and no k independent vertices.

Suppose G contains a K_k. That means there is a set system $\mathcal{S} \subset V(G)$, with ground set $X = [m]$, such that $|S \cap T| = 1$ for all $S, T \in \mathcal{S}$. Since also $|S| = 3$ for all $S \in \mathcal{S}$, this set system satisfies the conditions of Fisher's Inequality, so $k = |\mathcal{S}| \leq |X| = m$.
On the other hand, suppose there are k independent vertices (ie G contains a $\overline{K_k}$). Then there is a set system $\mathcal{S} \subset V(G)$, with ground set $X = [m]$, such that $|S \cap T| \in \{0, 2\}$ for all $S, T \in \mathcal{S}$, and $|S| = 3$ for all $S \in \mathcal{S}$. This satisfies the conditions of the Oddtown theorem, so again $k = |\mathcal{S}| \leq |X| = m$.
So if G contains a K_k or a $\overline{K_k}$, then $k \leq m$. Therefore G is an $(m+1)$-Ramsey graph, and it follows that $R(k) \geq (k-1)^3$.