1. Prove the following inequality for \( n \geq 1 \):

\[
 n! \geq e \cdot \frac{n^n}{e^n}.
\]

(Hint: \( \int \ln x \, dx = x \ln x - x + c \).)

2. A family \( \mathcal{F} \) of nonempty subsets of a finite set \( X \) is \textit{two-colorable} if there exists a coloring of the elements of \( X \) using two colors with no set in \( \mathcal{F} \) monochromatic. For an integer \( n \geq 1 \), let \( m(n) \) denote the minimum size of a non-two-colorable family of \( n \)-element subsets of a finite set. Prove that \( m(n) \geq 2^{n-1} \).

In other words, prove that any family of fewer than \( 2^{n-1} \) \( n \)-element subsets of a finite set is two-colorable.

3. A \textit{tournament} is a directed graph such that any two vertices are connected by exactly one directed edge. Prove that if \( \binom{n}{k}(1 - 2^{-k})^{n-k} < 1 \), then there exists a tournament on \( n \) vertices with the following property: for any set \( X \) of \( k \) vertices there is a vertex \( v \notin X \) such that all the \( k \) edges connecting it with \( X \) are directed towards \( v \).

In other words, for any set of \( k \) players there is another player that beats all of them.

4. A \textit{matching} in a graph is a set of edges no two of which share an endpoint. Let \( G \) be a graph with \( m \) edges and a matching of size \( \mu \). Prove that \( G \) has a bipartite subgraph with at least \( (m + \mu)/2 \) edges.

5. Let \( G \) be a graph with \( n \) vertices and \( nd/2 \) edges, where \( d \geq 1 \). Prove that \( \alpha(G) \geq n/(2d) \).

(Hint: Take a random subset of the vertices so that the difference between the expected number of vertices and the expected number of edges inside this subset is \( n/(2d) \).)

6. * Let \( m(n) \) be defined as in Problem 2. Prove that if there exists an even \( v \) with

\[
 2^v \left( 1 - \frac{2^{v/2}}{\binom{n}{v}} \right)^m < 1,
\]

then \( m(n) \leq m \).

Careful calculation of the above bound yields \( m(n) < (1 + o(1)) \epsilon \ln 2 \cdot n^2 2^{n-2} \).