6. * Let \( m(n) \) be defined as in Problem 2. Prove that if there exists \( v \) with
\[
2^v \left( 1 - \frac{2^{(v/2)}}{\binom{n}{v}} \right)^m < 1,
\]
then \( m(n) \leq m \).

Let \( X \) be a set of size \( v \). Let \( \mathcal{F} = \{S_1, \ldots, S_m\} \), where \( S_1, \ldots, S_m \) are \( n \)-element subsets of \( X \) chosen independently uniformly at random (note that we do not assume \( S_1, \ldots, S_m \) to be distinct). We show that with positive probability the family \( \mathcal{F} \) is not two-colorable.

Let \( \mathcal{C} \) denote the family of all colorings of \( X \) with two colors, say, red and blue. For a coloring \( c \in \mathcal{C} \) and \( i \in \{1, \ldots, m\} \), let \( A_i^c \) denote the event that the set \( S_i \) is monochromatic under the coloring \( c \). If \( r \) and \( b \) denote the numbers of elements colored red and blue, respectively (so that \( r + b = v \)), then we have
\[
P(A_i^c) = \frac{\binom{r}{n} + \binom{b}{n}}{\binom{n}{v}},
\]
as there are \( \binom{r}{n} \) choices of a red subset, \( \binom{b}{n} \) choices of a blue subset, and \( \binom{v}{n} \) choices in total. The sum \( \binom{r}{n} + \binom{b}{n} \) is minimized when \( r = b = \frac{v}{2} \), that is, we have \( \binom{r}{n} + \binom{b}{n} \geq 2^{(v/2)} \). Indeed, for any \( p, q, x, y \) such that \( p \geq q \) and \( x \geq y \) or \( p \leq q \) and \( x \leq y \) we have
\[
px + qy \geq px + qy - \frac{1}{2}(p - q)(x - y) = \frac{1}{2}(p + q)(x + y),
\]
which applied \( n \) times yields
\[
\begin{align*}
&\frac{r}{2}(r - 1) \ldots (r - n + 1) + b(b - 1) \ldots (b - n + 1) \\
&\geq \frac{r+b}{2}(r - 1) \ldots (r - n + 1) + \frac{r+b}{2}(b - 1) \ldots (b - n + 1) \\
&\geq \ldots \\
&\geq \frac{r+b}{2}(\frac{r+b}{2} - 1) \ldots (\frac{r+b}{2} - n + 1) + \frac{r+b}{2}(\frac{r+b}{2} - 1) \ldots (\frac{r+b}{2} - n + 1) \\
&= 2 \cdot \frac{1}{n} \left( \frac{v}{2} - 1 \right) \ldots \left( \frac{v}{2} - n + 1 \right), \\
&\binom{v}{n} + \binom{v}{n} \geq 2 \cdot \frac{1}{n} \left( \frac{v}{2} - 1 \right) \ldots \left( \frac{v}{2} - n + 1 \right) = 2^{(v/2)}. \\
\end{align*}
\]
Consequently, we have
\[
P(A_i^c) \geq \frac{2^{(v/2)}}{\binom{n}{v}},
\]
\[
P(\overline{A_i^c}) = 1 - P(A_i^c) \leq 1 - \frac{2^{(v/2)}}{\binom{n}{v}}.
\]
Let \( B_c \) denote the event that none of the sets \( S_1, \ldots, S_m \) is monochromatic under \( c \). Since the subsets are chosen independently, we have
\[
P(B_c) = \prod_{i=1}^m P(\overline{A_i^c}) \leq \left( 1 - \frac{2^{(v/2)}}{\binom{n}{v}} \right)^m.
\]
Since $|\mathcal{C}| = 2^v$ and by the union bound, we have

$$P\left(\bigcup_{c \in \mathcal{C}} B^c\right) \leq \sum_{c \in \mathcal{C}} P(B^c) \leq 2^v \left(1 - \frac{2^{(v/2)}}{n} \right)^m < 1.$$ 

Therefore, there is a way to select $S_1, \ldots, S_m$ so that none of the events $B_c$ happens, that is, for each coloring $c \in \mathcal{C}$, at least one of the sets $S_1, \ldots, S_m$ is monochromatic under $c$, that is, the family $\mathcal{F}$ is not two-colorable. Consequently, we have $m(n) \leq |\mathcal{F}| \leq m$. 