Advanced Discrete Mathematics 2013 – Problem Set 11

1. Prove that for every $r \in \mathbb{N}$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that every two-coloring of the edges of any graph with $n \geq n_0$ vertices and more than $(\frac{1}{2} - \alpha)n^2$ edges yields a monochromatic copy of K_r.

   (Hint: To obtain a monochromatic copy of K_r, first apply Turán's theorem to get a large complete subgraph, and then apply Ramsey's theorem on this subgraph.)

2. (Chvátal-Rödl-Szemerédi-Trotter theorem) Prove that for every $\Delta \in \mathbb{N}$ there is a (big) constant c such that for any graph H with $\Delta(H) \leq \Delta$, every two-coloring of the edges of $K_{c \cdot v(H)}$ yields a monochromatic copy of H.

   (Hint: Suppose given a two-coloring of the edges of K_n with colors red and blue, for n large enough.

   (1) Apply Szemerédi's regularity lemma to the graph G formed by the red edges.

   (2) Build the regularity graph R with edges representing all ε-regular pairs (of any density).

   (3) Color the edges of R according to the density of ε-regular pairs: red if the density is $\geq \frac{1}{2}$, blue otherwise.

   (4) Apply the result from problem 1 to find in R a monochromatic copy of $K_{\Delta+1}$.

   (5) Using the fact that $\chi(H) \leq \Delta + 1$ and the general embedding lemma, show that G or $\overline{G}$ contains a copy of H.

   In your proof, choose all constants appropriately as required for all of the steps.)

3. (Alon-Yuster theorem) Prove that for every graph H and every $\alpha > 0$ there is $n_0 \in \mathbb{N}$ such that if G is a graph with $n \geq n_0$ vertices and with minimum degree

$$\delta(G) \geq \left(\frac{\chi(H) - 1}{\chi(H)} + \alpha \right) n,$$

   then there are $(1 - \alpha)n/v(H)$ vertex-disjoint copies of H in G. Use the following *Hajnal-Szemerédi theorem*: *If G is a graph with $\delta(G) \geq \frac{r-1}{r}n$, then G contains $\lfloor n/r \rfloor$ vertex-disjoint copies of K_r.*

   (Hint: Suppose given a large enough graph G.

   (1) Apply Szemerédi's regularity lemma to G.

   (2) Build the regularity graph R with edges representing ε-regular pairs of density at least d.

   (3) Apply Hajnal-Szemerédi theorem to find an almost-covering of R by vertex-disjoint copies of $K_{\chi(H)}$.

   (4) Use the general embedding lemma repeatedly to cover the corresponding $\chi(H)$-tuples of partition sets almost completely by copies of H.

   In your proof, choose all constants appropriately as required for all of the steps.)

4. * Let (A, B) be an ε-regular bipartite graph with $|A| = |B| = n$ and with density $d > 2\varepsilon$. Prove that (A, B) contains a path with at least $2(1 - \frac{\varepsilon}{d-\varepsilon})n$ vertices.

   (Hint: Apply a similar embedding procedure as in the proof of the general embedding lemma.)

5. * Prove that for any $p \in (0, 1]$ and $\alpha > 0$, an n-vertex random graph G with edge probability p has the following property with probability approaching 1 as $n \to \infty$: every subgraph of G with at least $(\frac{1}{4}p + \alpha)n^2$ edges contains a triangle. Use the following *Chernoff's bound*: *For any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that if $X_1, \ldots, X_n$ are independent random variables with values in $\{0, 1\}$ and $\mu = E(X_1 + \ldots + X_n)$, then*

$$P(|X_1 + \ldots + X_n - \mu| > \varepsilon\mu) < 2e^{-c_\varepsilon \mu}.$$

   (Hint: First, show that with probability approaching 1 as $n \to \infty$ any pair (A, B) in G with $|A| = |B| \geq \beta n$ satisfies $d(A, B) \leq p + \varepsilon$, for appropriate constants ε and β. Then, take any subgraph $\tilde{G}$ of such G with at least $(\frac{1}{4}p + \alpha)n^2$ edges, apply Szemerédi's regularity lemma to $\tilde{G}$, take the regularity graph R representing ε-regular pairs with densities $\geq 2\varepsilon$, show that R contains a triangle, and use the triangle counting lemma to conclude that $\tilde{G}$ contains a triangle.)