1. Prove the following facts from linear algebra (over $\mathbb{R}$):

(a) \(\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)\) for any two \(m \times n\) matrices \(A, B\);

(b) \(\text{rk}(AB) \leq \min (\text{rk}(A), \text{rk}(B))\) for any \(k \times l\) matrix \(A\) and \(l \times m\) matrix \(B\);

(c) if an \(n \times n\) matrix \(M\) is positive definite, then \(\text{rk}(M) = n\).

2. In the proof of Fisher’s Inequality we used the fact that any matrix of the form \(bJ + D\) is nonsingular, if \(b \geq 0\) and \(D\) is a diagonal matrix with diagonal entries \(> 0\). For example this one, if all \(a_i > b \geq 0\):

\[
\begin{pmatrix}
a_1 & b & b & b \\
b & a_2 & b & b \\
b & b & a_3 & b \\
b & b & b & a_4
\end{pmatrix}
\]

Prove this in 3 ways: using positive definiteness, the determinant, and the definition of linear independence.

3. Given a bound on the size of a certain kind of object, a \textit{tight example} is such an object with size exactly equal to that bound.

Give two families of tight examples for the bound for 1-intersecting set systems.

In other words, give two constructions that for any \(|X|\) give two tight examples for \(|S| \leq |X|\) (which are really different, ie not the same after relabelling). For convenience take \(X = \{1, 2, \ldots, n\}\). Note that we are not requiring uniformity.

4. Given \(n\) points in \(\mathbb{R}^2\) which are not collinear, prove that there are at least \(n\) lines that pass through at least two of these points.

(Hint: Compare the fact that there is exactly one line through any two points with the condition of Fisher’s Inequality.)

5. Show that there are at least \(2^{n-4}\) different ways to decompose \(K_n\) into \(n - 1\) bicliques.

Again they should be really different, ie not the same after relabelling.

6. Suppose we have sets \(A_1, \ldots, A_{n+1} \subset X\) and \(|X| = n\). Show that there are two disjoint sets \(I, J\) of indices such that

\[
\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.
\]

7. * Given \(|X| = n\) and distinct nonempty subsets \(A_1, \ldots, A_n \subset X\). Show that there is an \(x \in X\) such that the sets \(A_i \setminus \{x\}\) are still distinct.

8. * The examples that you found in problem 3 are not uniform. Show that there exists a tight example for the bound \(|S| \leq |X|\) for \textit{uniform} 1-intersecting set systems in the case \(|X| = 7\).

Show that in general this is only possible if \(|X| = q^2 + q + 1\) for some integer \(q > 0\).