Exercises Week 3

1. In the following graph find a maximum matching and a minimum vertex cover.

![Graph Image]

**Definition 1.** For a graph $G$ we say that $M$ is a perfect matching of $G$ if the number of edges of $M$ is equal to $|V(G)|/2$.

2. Prove that in a tree, there is at most one perfect matching.

3. Show that a $k$-regular (a graph where every vertex has degree equal to $k$) bipartite graph is a union of $k$ perfect matchings that do not share an edge.

4. Prove that in a bipartite graph $G$ the sum of the maximum number of edges in a matching with the maximum cardinality of an independent set is $|V(G)|$.

5. Prove the following theorem without using Gallai and Milgram’s theorem: For every graph $G$ there is a path cover $\mathcal{P}$ and an independent set $I$ of size $|\mathcal{P}|$ that has exactly one vertex from each path in $\mathcal{P}$.

6. Prove Konig’s theorem using Gallai and Milgram’s theorem.

7. Let $a_1, a_2, ..., a_n$ be a sequence of distinct real numbers. Show that there is either an increasing subsequence or a decreasing subsequence of size at least $\sqrt{n}$.

**Bonus Problem:** Find a partial ordered set that has no infinite antichain but is not a union of finitely many chains.

Given out: Thursday, March 5