Exercises Week 11

1 Understanding the definitions

1. Show that \(R(2, t) = t\).

2. What is the expectation of the number of edges in \(G(n, 1/2)\)?

3. Does graph \(G(n, 1/2)\) has more than \(\binom{n}{2}/2\) edges with positive probability?

4. What is the probability that a random graph in \(G(n, p)\) has exactly \(m\) edges, for \(0 \leq m \leq \binom{n}{2}\) fixed? At most \(m\) edges?

2 Exercises

1. For every \(s \in \mathbb{N}\) there exists a constant \(c\) such that every graph on \(n\) vertices with no \(K_{s,2}\) graph has at most \(cn^{3/2}\) edges.

   **Definition.** For graphs \(H_1\) and \(H_2\) we define \(R(H_1, H_2)\) to be the smallest integer \(n\) such that for every graph \(G\) with \(n\) vertices either \(H_1\) is a subgraph of \(G\) or \(H_2\) is a subgraph of \(\overline{G}\).

2. Show that \(R(K_3, K_3) = 6\).

   Note: in the lecture notations, \(R(K_3, K_3)\) is \(R(3, 3)\).

3. Calculate \(R(P_3, K_3)\) and \(R(P_4, P_4)\) (\(P_k\) is a path with \(k\) vertices).

4. a) Prove that in a graph with minimal degree \(t - 1\) one can find any tree on \(t\) vertices as a subgraph.

   b) Prove that \(R(T, K_s) = (s - 1)(t - 1) + 1\) for any tree \(T\) on \(t\) vertices.

5. What is the expected number of \(K_r\)-subgraphs in \(G \in \mathcal{G}(n,p)\)? What about \(K_{1,3}\)-subgraphs in \(G \in \mathcal{G}(n,p)\)?

**Bonus Problem:** Prove that no matter how we color the natural numbers \(\mathbb{N}\) into two colours, there will be a monochromatic 3-term arithmetic progression.

Given out: Thursday, May 21