Homomorfism to discrete Borsuk graph

February 11, 2009

Let $G_{n,k}(V,E)$, $k \leq n$, denote discrete Borsuk graph with the set of vertices $V(G_{n,k}) = \{+1,-1\}^n$ and the set of edges $E(G_{n,k}) = \{uv|\ d_H(u,v) \geq n-k\}$. Thus, a vertex $v \in V(G_{n,k})$ is represented by a string $v_1 \dots v_n$, $v_i \in \{+1,-1\}$, of length n, and two vertices are joined with an edge if their Hamming distance is at least $n-k$.

Let $G'_{n,k}(V,E)$, $k \leq n$, denote a graph with the set of vertices $V(G'_{n,k}) = \{+1,-1,+2,-2\}^n$ and the set of edges $E(G'_{n,k}) = \{uv|\ \forall i\ \{u_i,v_i\} \in \{\{+1,-1\},\{+1,+1\},\{-1,-1\},\{+2,-1\},\{-2,+1\},$ $\{-2,+2\}\}$ and $|\{i|\ \{u_i,v_i\} = \{+2,-2\}\}| \geq n-k\}$. Thus, a vertex $v \in V(G)$ is represented by a string $v_1 \dots v_n$, but now $v_i \in \{+1,-1,+2,-2\}$, of length n, and two vertices are joined with an edge if they are opposite in some sense and if the number of "ambiguities" is at most k.

Our problem can be stated as follows: Is there a graph homomorphism h from $V(G'_{n,l})$ to $V(G_{n,k})$, if $l > k+1$, for infinitely many n (if we consider k and l to be fixed) ? If that is the case try to find the largest such l in terms of k.

In what follows that we show that it is doable in the case when $l = k+1$ and if $n-k \equiv 0 (\bmod 2)$.

We construct a graph homomorphism h from $V(G'_{n,k+1})$ to $V(G_{n,k})$, if $n-k \equiv 0 (\bmod 2)$ and $k > 0$ (the case $k=0$ is trivial). Let $\mathrm{par}(v) = \sum_{i=1}^{n} \mathrm{sgn}(v_i) \mod 2$, $v \in V(G'_{n,k})$ or $v \in V(G_{n,k})$, where $\mathrm{sgn}(a) = 1$, if $a > 0$, and $\mathrm{sgn}(a) = 0$, otherwise. Thus, $\mathrm{par}(v)$ is a parity function. Let $c(a) = 1$, if $a > 0$, and $c(a) = -1$, otherwise.

$$h(v) = \begin{cases} c(v_1)\dots c(v_n) & |\{i|\ v_i \in \{+1,-1\}\}| = 0 \\ c(v_1)\dots c(v_n) & |\{i|\ v_i \in \{+1,-1\}\}| > 0 \quad \text{and} \quad \mathrm{par}(v) = 0 \\ c(v_1)\dots - c(v_j)\dots c(v_n) & |\{i|\ v_i \in \{+1,-1\}\}| > 0 \quad \text{and} \quad \mathrm{par}(v) = 1 \\ & v_j \in \{+1,-1\} \text{ (chosen arbitrarily)} \end{cases}$$

In order to prove that h is a homomorphism we show that if $uv \in E(G'_{n,k+1})$ then $h(u)h(v) \in E(G_{n,k})$. If u or v does not contain $+1$ or -1 we are done, since now, $d_H(h(u)h(v)) \geq n-1$ and $k > 0$. The situation is also trivial if $|\{i|\ \{u_i,v_i\} = \{+2,-2\}\}| > n-k-1$. So, from now on we assume that both u and v contain at least one $+1$ or -1, and $|\{i|\ \{u_i,v_i\} = \{+2,-2\}\}| = n-k-1$. We have constructed h such that the vertices having at least one 1 are mapped to the vertices w with $\mathrm{par}(w) = 0$. Hence, if $uv \in E(G'_{n,k+1})$ and $|\{i|\ \{u_i,v_i\} = \{+2,-2\}\}| = n-k-1$, the mapped ambiguous parts of v and u (i.e. all $c(v_i)$ and $c(u_i)$ such that $\{u_i,v_i\} \neq \{+2,-2\}$) cannot be equal at all positions, as otherwise $\mathrm{par}(h(u)) \neq \mathrm{par}(h(v))$ (note that $n-k-1 \equiv 1$). Thus, $d_H(h(u)h(v)) \geq n-k$, and that in turn implies that $h(u)h(v) \in E(G_{n,k})$.