An improvement on the number of simplices in $\mathbb{F}_q^d$

Duc Hiep Pham    Thang Pham    Le Anh Vinh

Abstract

Let $E$ be a set of points in $\mathbb{F}_q^d$. Bennett, Hart, Iosevich, Pakianathan, and Rudnev (2016) proved that if $|E| \gg q^{d-k^2+1}$ then $E$ determines a positive proportion of all $k$-simplices. In this paper, we give an improvement of this result in the case when $E$ is the Cartesian product of sets. Namely, we show that if $E$ is the Cartesian product of sets and $q^{kd(k+1)+1}/d = o(|E|)$, the number of congruence classes of $k$-simplices determined by $E$ is at least $(1-o(1))q^{(k+1)/2}$, and in some cases our result is sharp.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of order $q$ with $q = p^r$ for some prime $p$ and positive integer $r$. Denote by $O(d, \mathbb{F}_q)$ the orthogonal group in $\mathbb{F}_q^d$. We say that two $k$-simplices in $\mathbb{F}_q^d$ with vertices $(x_1, \ldots, x_{k+1})$, $(y_1, \ldots, y_{k+1})$ are in the same congruence class if there exist $\theta \in O(d, \mathbb{F}_q)$ and $z \in \mathbb{F}_q^d$ so that $z + \theta(x_i) = y_i$ for all $i = 1, 2, \ldots, k+1$.

Hart and Iosevich [8] made the first investigation on counting the number of congruence classes of simplices determined by a point set in $\mathbb{F}_q^d$. More precisely, they proved that if $|E| \gg q^{k^2+1}$ with $d \geq (k+1)/2$, then $E$ contains a copy of all $k$-simplices with non-zero edges. Here and throughout, $X \ll Y$ means that there exists $C > 0$ such that $X \leq CY$, and $X = o(Y)$ means that $X/Y \to 0$ as $q \to \infty$, where $X, Y$ are viewed as functions in $q$.

Using methods from spectral graph theory, the third listed author [13] improved this result. In particular, he showed that the same result also holds when $d \geq 2k$ and $|E| \gg q^{d-1/2+k}$. It follows from the results in [8, 13] that the most difficulties arise when the size of simplex is large with respect to the dimension of the space, for instance, the result in [13] on the number of congruence classes of triangles is only non-trivial if $d \geq 4$.

In [4], Covert et al. addressed the case of triangles in $\mathbb{F}_q^2$, and they established that if $|E| \gg \rho q^2$, then $E$ determines at least $c \rho q^3$ congruence classes of triangles. The author of [12] extended this result to the case $d \geq 3$. Formally, he proved that if $|E| \gg q^{d/2}$, then $E$ determines a positive proportion of all triangles. Using Fourier analytic techniques, Chapman et al. [5] indicated that the threshold $q^{d/2}$ on the cardinality of $E$ in the triangle case can be replaced by $q^{d/2}$ for the case of $k$-simplices. In a recent result, Bennett et al. [3] improved the threshold $q^{d/2}$ to $q^{d-k^2+1}$. The precise statement is given by the following theorem.
Theorem 1.1 (Bennett et al., [3]). Let $\mathcal{E}$ be a subset in $\mathbb{F}_q^d$. Suppose that
\[ |\mathcal{E}| \gg q^{d - \frac{d-1}{k+1}}, \]
then, for $1 \leq k \leq d$, the number of congruence classes of $k$-simplices determined by $\mathcal{E}$ is at least $c q^{(k+1)2}$ for some positive constant $c$.

Note that one of the most important steps in their proof in [3] is to reduce the problem of counting congruence classes of $k$-simplices to the number of quadruples $(a, b, c, d) \in \mathcal{E}^4$ with $||a - b|| = ||c - d||$ by applying Lemma 2.3 and elementary results from group action theory in an ingenious way. It has been shown in [3] that the number of such quadruples in $\mathcal{E}^4$ is at most $|\mathcal{E}|^4/q + q^d|\mathcal{E}|^2$. We remark here that the error term $q^d|\mathcal{E}|^2$ plays an important role in their arguments when $\mathcal{E}$ is large enough. We refer the reader to Section 5 for a detailed explanation and a discussion on a connection between Fourier analytic techniques and methods from spectral graph theory.

In this paper, by employing spectral graph theory techniques, we are able to get a better estimate on the number of quadruples $(a, b, c, d) \in \mathcal{E}^4$ with $||a - b|| = ||c - d||$ in the case $\mathcal{E}$ is a Cartesian product of sets. This allows us to obtain the following improvement of Theorem 1.1.

Theorem 1.2. Let $\mathcal{E} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_d$ be a subset in $\mathbb{F}_q^d$. Suppose that
\[ \left( \min_{1 \leq i \leq d} |\mathcal{A}_i| \right)^{-1} |\mathcal{E}|^{k+1} \gg q^{kd}, \]
then for $1 \leq k \leq d$, the number of congruence classes of $k$-simplices determined by $\mathcal{E}$ is at least $c q^{(k+1)/2}$ for some positive constant $c$.

Corollary 1.3. Let $\mathcal{E} = \mathcal{A}_d$ be a subset in $\mathbb{F}_q^d$. If $|\mathcal{E}| \gg q^{\frac{kd}{k+1}}$ then the number of congruence classes of $k$-simplices determined by $\mathcal{E}$ is at least $c q^{(k+1)/2}$ for some positive constant $c$.

As a consequence of Corollary 1.3, we recover the following result in [6].

Theorem 1.4. Let $\mathcal{A}$ be a subset in $\mathbb{F}_q$. If $|\mathcal{A}| \gg q^{\frac{d}{d-1}}$, then the number of distinct distances determined by points in $\mathcal{A}^d \subseteq \mathbb{F}_q^d$ is at least $\gg q$.

On the number of congruence classes of triangles in $\mathbb{F}_q^2$. For the case of triangles in $\mathbb{F}_q^2$, in 2012 Bennett, Iosevich, and Pakianathan [2], using Elekes-Sharir paradigm and an estimate on the number of incidences between points and lines in $\mathbb{F}_q^2$, improved significantly the result in [4]. In particular, they proved that if $|\mathcal{E}| \gg q^{7/4}$ and $q \equiv 3 \mod 4$, then the number of triangles determined by $\mathcal{E}$ is at least $c q^3$ for some positive constant $c$. The authors of [3] recently improved the exponent $7/4$ to $8/5$ in the following.

Theorem 1.5 (Bennett et al. [3]). Let $\mathcal{E}$ be a set of points in $\mathbb{F}_q^2$. If $|\mathcal{E}| \gg q^{8/5}$, then $\mathcal{E}$ determines a positive proportion of all triangles.
We will give a graph-theoretic proof for this theorem in Section 4. If \( \mathcal{E} \) has Cartesian product structure of sets with different sizes, as a consequence of Theorem 1.2, we are able to obtain a much stronger result as follows.

**Theorem 1.6.** Let \( \mathcal{A}, \mathcal{B} \) be subsets in \( \mathbb{F}_q \). If \( |\mathcal{A}| \geq q^{\frac{1}{2}+\epsilon} \) and \( |\mathcal{B}| \geq q^{1-\frac{2}{3}} \) for some \( \epsilon \geq 0 \), then the number of congruence classes of triangles determined by \( \mathcal{A} \times \mathcal{B} \subseteq \mathbb{F}_q^2 \) is at least \( cq^3 \) for some positive constant \( c \).

Note that if \( \mathcal{A} \) and \( \mathcal{B} \) are arbitrary sets in \( \mathbb{F}_q \), then it follows from Theorem 1.2 that in order to prove that there exist at least \( cq^3 \) congruence classes of triangles, we need the condition \( |\mathcal{A}|^2|\mathcal{B}|^3 \gg q^4 \). In particular, if \( |\mathcal{A}| < q^{1/2} \) then we must have \( |\mathcal{B}| > q \). In fact, one can not expect to get a positive proportion of congruence of triangles in the set \( \mathcal{A} \times \mathcal{B} \) with arbitrary sets \( \mathcal{A} \) and \( \mathcal{B} \) satisfying \( |\mathcal{A}| = o(q^{1/2}) \) and \( |\mathcal{B}| < q \), since the authors of [3] gave a construction with \( |\mathcal{A}| = q^{1/2-\epsilon} \) and \( |\mathcal{B}| = q \), and the number of congruence classes triangles determined by \( \mathcal{E} \) is at most \( cq^{3-\epsilon} \) for \( \epsilon > 0 \). For the sake of completness, we reproduce their construction here:

**A construction for the triangle case:** Let \( \epsilon \in (0, 1/2) \) and \( \tau \) be a non-null vector. We use the orthogonal decomposition \( \tau_1 \perp < \tau > \) to identify \( \mathbb{F}_q^2 = \mathbb{F}_q \oplus \mathbb{F}_p \). Suppose that \( q = p^r \) for some prime \( p \), so \( \mathbb{F}_q \) can be viewed as a \( r \)-dimensional space over \( \mathbb{F}_p \). Thus for any \( y \in \mathbb{F}_q \), we can write \( y \) as \( (y(1), \ldots, y(r)) \in \mathbb{F}_p^r \) where \( y(i) \) are the \( \mathbb{F}_p \) coordinates of \( y \in \mathbb{F}_q \) with respect to a \( \mathbb{F}_p \)-basis.

We now consider the following set

\[
\mathcal{E} = \{ (x, y) : x \in \mathbb{F}_q, 0 \leq y(i) \leq p^{1/2-\epsilon}, 1 \leq i \leq r \} \subseteq \mathbb{F}_q \oplus \mathbb{F}_q = \mathbb{F}_q^2.
\]

Let \( \mathcal{G} = \{(x_1, y_1), (x_2, y_2), (x_3, y_3) : (x_1, y_1) \in \mathcal{E}, -p^{1/2-\epsilon} \leq y_1(j) - y_i(j) \leq p^{1/2-\epsilon} \} \)

where \( 1 \leq i \leq 3, 1 \leq j \leq r \).

Notice that we can view \( \mathcal{G} \) as a set of triangles and \( \mathcal{E}^3 \subseteq \mathcal{G} \). This implies that the number of distinct congruence classes of triangles in \( \mathcal{E} \) is smaller than that of \( \mathcal{G} \). Moreover if we translate a triangle in \( \mathcal{E} \) by a vector in \( -\mathcal{E} \), that triangle will still be in \( \mathcal{G} \). Therefore any triangle of \( \mathcal{E} \) occurs in \( \mathcal{G} \) with multiplicity at least \( |\mathcal{E}| \). In other words, any congruence class of triangle that appears in \( \mathcal{G} \) occurs at least \( |\mathcal{E}| \) times. It follows from \( \mathcal{E} \)'s construction that \( |\mathcal{E}| = q^{3/2-\epsilon} \), and the number of congruence classes of triangles in \( \mathcal{E} \) is at most

\[
\frac{|\mathcal{G}|}{|\mathcal{E}|} \leq 2^r |\mathcal{E}| (2^r |\mathcal{E}|)^2 = 2^{3r} |\mathcal{E}|^2 = 2^{3r} q^{3-2\epsilon}.
\]

One can check that \( 2^{3r} = q^{3 \log_q(2)} \). Thus when \( p \to \infty \), we can choose \( 2\epsilon > 3 \log_p(2) \). In conclusion, the number of congruence classes of triangles in \( \mathcal{E} \) is \( o(q^3) \).

This construction can also be extended to arbitrary \( k \) and \( d \) with \( k \leq d \), see [3, Section 5] for more details.
Let \(k\) and \(d\) be integers with \(k \leq d\), and \(\alpha_{k,d}\) be the infimum of numbers \(s > 0\) such that if \(|E| \gg q^s\) then the number of congruence classes of \(k\)-simplices in \(E\) is \(cq^{\frac{k+1}{2}}\) for some positive constant \(c\). The authors of [3] gave the following theorem on lower bounds of \(\alpha_{k,d}\).

**Theorem 1.7 (Bennett et al.,[3]).** For integers \(k\) and \(d\) with \(k \leq d\),

i. if \(k = 1\) and \(d \geq 3\) odd, we have \(\alpha_{1,d} \geq \frac{d+1}{2}\).

ii. if \(k = 1\) and \(d \geq 2\) is even, we have \(\alpha_{1,d} \geq \frac{d}{2}\).

iii. if \(k > 1\), we have \(\alpha_{k,d} \geq k - 1 + \frac{1}{k}\).

**Distinct distance subsets in \(\mathbb{F}_q^d\).** Given a subset \(E \subset \mathbb{F}_q^d\), a subset \(U \subset E\) is called a **distinct distance subset** if there are no four distinct points \(x, y, z, t \in U\) such that \(||x - y|| = ||z - t||\). In [10], Phuong et al. studied the finite field analogue of this problem. More precisely, the authors of [10] proved that if \(|E| \geq 2q^{2d+1/3}\), then there exists a distinct distance subset of cardinality \(\gg q^{1/3}\). This implies that the result is only non-trivial when \(d \geq 3\). In this paper, we fill in this gap. In particular, we prove that if \(d = 2\), then the threshold \(q^{(2d+1)/3}\) can be improved to \(q^{4/3}\). In particular, the statement is in the following.

**Theorem 1.8.** Let \(E\) be a subset in \(\mathbb{F}_q^2\) with \(|E| \gg q^{4/3}\). There exists a distinct distance subset \(U \subset E\) satisfying \(q^{1/3} \ll |U| \ll q^{1/2}\).

The rest of this paper is organized as follows: in Section 2 we mention some graph-theoretic tools; Section 3 contains a proof of Theorem 1.2; the proofs of Theorems 1.5 and 1.8 are mentioned in Section 4; we end this paper with some discussions in Section 5.

## 2 Tools from spectral graph theory

A graph \(G = (V, E)\) is called an \((n, d, \lambda)\)-graph if it is \(d\)-regular, has \(n\) vertices, and the second eigenvalue of \(G\) is at most \(\lambda\). The result below gives an estimate on the number of edges between two multi-sets of vertices in an \((n, d, \lambda)\)-graph.

**Lemma 2.1 ([7]).** Let \(G = (V, E)\) be an \((n, d, \lambda)\)-graph. For any two multi-sets of vertices \(B, C\), we have

\[
\left| e(B, C) - \frac{d|B||C|}{n} \right| \leq \lambda \left( \sum_{b \in B} m_B(b)^2 \right)^{1/2} \left( \sum_{c \in C} m_C(c)^2 \right)^{1/2},
\]

where \(e(B, C)\) is the number of edges between \(B\) and \(C\) in \(G\), and \(m_X(x)\) is the multiplicity of \(x\) in \(X\).
Let $PG(q,d)$ denote the projective geometry of dimension $d - 1$ over finite field $\mathbb{F}_q$. The vertices of $PG(q,d)$ correspond to the equivalence classes of the set of all non-zero vectors $[x] = [x_1, \ldots, x_d]$ over $\mathbb{F}_q$, where two vectors are equivalent if one is a multiple of the other by a non-zero element of the field.

In this section, we recall a well-known construction of the Erdős-Rényi graph due to Alon and Krivelevich [1] as follows. Let $ER(\mathbb{F}_d^d)$ denote the graph whose vertices are the points of $PG(q,d)$, and two (not necessarily distinct) vertices $[x] = [x_1, \ldots, x_d]$ and $[y] = [y_1, \ldots, y_d]$ are adjacent if and only if $x_1y_1 + \ldots + x_dy_d = 0$. Alon and Krivelevich [1] obtained the following result on the spectrum of $ER(\mathbb{F}_d^d)$.

Lemma 2.2 (Alon and Krivelevich, [1]). For any odd prime power $q$ and $d \geq 2$, the Erdős-Rényi graph $ER(\mathbb{F}_d^d)$ is an $(\frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}, \frac{q^{(d-2)/2}}{q-1})$-graph.

The next lemma is useful in the proof of Theorem 1.2, which allows us to reduce $k$-simplices to 2-simplices.

Lemma 2.3 (Bennett et al., [3]). Let $V$ be a finite space and $f: V \to \mathbb{R}_{\geq 0}$ a function. For any $n \geq 2$ we have

$$\sum_{z \in V} f^n(z) \leq |V| \left( \frac{\|f\|_1}{|V|} \right)^n + \frac{n(n-1)}{2} \|f\|_\infty^{n-2} \sum_{z \in V} (f(z) - \frac{\|f\|_1}{|V|})^2,$$

where $\|f\|_1 = \sum_{z \in V} |f(z)|$, and $\|f\|_\infty = \max_{z \in V} f(z)$.

3 Proof of Theorem 1.2

For $E = A_1 \times \cdots \times A_d \subseteq \mathbb{F}_q^d$ and $t \in \mathbb{F}_q$, we define $\nu_E(t)$ as the cardinality of the set $\{(x,y) \in E \times E : ||x - y|| = t\}$. In order to prove Theorem 1.2 we first need the following lemma.

Lemma 3.1. For $E = A_1 \times \cdots \times A_d \subseteq \mathbb{F}_q^d$ with $|A_d| = \min_{1 \leq i \leq d} |A_i|$, $|E| = \sum_{t \in \mathbb{F}_q} \nu_E(t)$.

$$\sum_{t \in \mathbb{F}_q} \nu_E(t)^2 \leq \frac{|E|^4}{q} + 2q^{d-1}|E|^2|A_d|.$$

Proof. For a fixed pair $(a, b) \in A_d^2$, let $N(a, b)$ denote the set of quadruples $(x, y, z, t) \in E^4$ with $x = (x_1, \ldots, x_{d-1}, a)$ and $y = (y_1, \ldots, y_{d-1}, b)$ satisfying $||x - z|| = ||y - t||$. Then one has

$$\sum_{t \in \mathbb{F}_q} \nu_E(t)^2 = \sum_{(a, b) \in A_d^2} |N(a, b)| \leq |A_d|^2 \max_{(a, b) \in A_d^2} |N(a, b)|.$$
We next show that
\[ |N(a, b)| \leq \frac{|E|^2 |A_d|^{-2}}{q} + q^{d-1} |E||A_d|^{-1}, \quad \forall (a, b) \in A_d. \]

Indeed, let \( U \) and \( V \) be, respectively, multi-subsets in \( PG(q, 2d) \) defined by
\[
U = \left\{ \left[ -2x_1, \ldots, -2x_{d-1}, 1, t_1, \ldots, t_{d-1}, -(t_d - b)^2 - \sum_{i=1}^{d-1} t_i^2 + \sum_{i=1}^{d-1} x_i^2 \right] : x_i, t_i \in A_i \right\},
\]
and
\[
V = \left\{ \left[ z_1, \ldots, z_{d-1}, (z_d - a)^2 + \sum_{i=1}^{d-1} z_i^2 - \sum_{i=1}^{d-1} y_i^2, 2y_1, \ldots, 2y_{d-1}, 1 \right] : z_i, t_i \in A_i \right\}.
\]

It is clear that
\[ |U| = |E||A_d|^{-1}, \quad |V| = |E||A_d|^{-1}, \quad m_U(u) \leq 2, \quad m_V(v) \leq 2, \quad \forall u \in U, v \in V. \]

and \( |N(a, b)| \) is equal to the number of edges between \( U \) and \( V \) in the Erdős-Rényi graph \( \mathcal{E}R(F_q^{2d}) \). Thus it follows from Lemmas 2.1 and 2.2 that
\[ |N(a, b)| < \frac{|E|^2 |A_d|^{-2}}{q} + 2q^{d-1} |E||A_d|^{-1}, \]
and this completes the proof of the lemma.

It is convenient to recall the following definition which is given in [3].

**Definition 3.2.** Let \( V \) be the \( F_q \)-vector space of \((k + 1) \times (k + 1)\) symmetric matrices \( \mathbb{D} \) which can be viewed as the space of possible ordered \( k \)-simplex distances. For \( \mathcal{E} \subset F_q^d \), we define \( \mu : V \to \mathbb{Z} \)
\[
\mu(\mathbb{D}) := \# \left\{ (x_1, \ldots, x_{k+1}) \in \mathbb{E}^{k+1} : ||x_i - x_j|| = d_{i,j}, 1 \leq i < j \leq k + 1 \right\}
\]

**Proof of Theorem 1.2.** Suppose that \( |A_d| = \min_{1 \leq i \leq d} |A_i| \). Let \( T_{k,d}(\mathcal{E}) \) denote the set of congruence classes of \( k \)-simplices determined by \( \mathcal{E} \). It follows from the Cauchy-Schwarz inequality that
\[
\sum_{\mathbb{D}} \mu(\mathbb{D}) \leq \left( \sum_{\mathbb{D} \in \text{supp}(\mu)} 1 \right)^{1/2} \left( \sum_{\mathbb{D} \in \text{supp}(\mu)} \mu(\mathbb{D})^2 \right)^{1/2} = |T_{k,d}(\mathcal{E})|^{1/2} \left( \sum_{\mathbb{D} \in \text{supp}(\mu)} \mu(\mathbb{D})^2 \right)^{1/2}.
\]
This gives
\[
|T_{k,d}(\mathcal{E})| \geq \frac{(\sum_{\mathbb{D}} \mu(\mathbb{D}))^2}{\sum_{\mathbb{D}} \mu(\mathbb{D})^2} \geq \frac{|\mathcal{E}|^{2k+2}}{\sum_{\mathbb{D}} \mu(\mathbb{D})^2}. \]
For $\theta \in O(d, \mathbb{F}_q)$ and $z \in \mathbb{F}_q^d$, we define

$$w_\theta(z) := \{(u, v) \in \mathcal{E}^2 : \theta(u) + z = v\}.$$ 

and denote the common stabilizer size of $k$-simplices in the congruence class $\mathcal{D}$ by $s(\mathcal{D})$.

It has been shown in [3] that $s(\mathcal{D}) \leq |O(d - k, \mathbb{F}_q)|$, and $|O(n, \mathbb{F}_q)| = 2(1 + o(1))q^{(n)}$. Furthermore, it is easy to check that

$$\sum_{\mathcal{D}} s(\mathcal{D}) \mu(\mathcal{D})^2 \leq \sum_{\theta \in O(d, \mathbb{F}_q), z \in \mathbb{F}_q^d} |w_\theta(z)|^{k+1},$$

where $|w_\theta(z)|$ is the cardinality of $w_\theta(z)$.

For a fixed $\theta$, it follows from Lemma 2.3 with $f(z) := |w_\theta(z)|$, $||f||_1 = |\mathcal{E}|$, and $||f||_{\infty} \leq |\mathcal{E}|$ that

$$\sum_{z \in \mathbb{F}_q^d} |w_\theta(z)|^{k+1} \leq |O(d, \mathbb{F}_q)| \frac{|\mathcal{E}|^{2k+2}}{q^{kd}} + \frac{k(k - 1)}{2} |\mathcal{E}|^{k-1} \sum_{\theta, z} \left( |w_\theta(z)| - \frac{|\mathcal{E}|^2}{q^d} \right)^2.$$ \hspace{1cm} (3.2)

Thus we obtain

$$\sum_{\theta \in O(d, \mathbb{F}_q), z \in \mathbb{F}_q^d} |w_\theta(z)|^{k+1} \leq |O(d, \mathbb{F}_q)| \frac{|\mathcal{E}|^{2k+2}}{q^{kd}} + \frac{k(k - 1)}{2} |\mathcal{E}|^{k-1} \sum_{\theta, z} \left( \sum_{\theta, z} |w_\theta(z)|^2 - \frac{|\mathcal{E}|^4 O(d, \mathbb{F}_q)}{q^d} \right).$$

It follows from the definition of $w_\theta(z)$ that $|w_\theta(z)|^2$ is equal to the number of quadruples $(a, b, c, d) \in \mathcal{E}^4$ satisfying $\theta(a) + z = c$ and $\theta(b) + z = d$. This implies that $\theta(a - b) = (c - d)$, and $||a - b|| = ||c - d||$. Since the stabilizer of a non-zero element in $\mathbb{F}_q^d$ is at most $|O(d - 1, \mathbb{F}_q)|$, it follows that each quadruple $(a, b, c, d) \in \mathcal{E}^4$, which satisfies $a - b \neq 0$, $||a - b|| = ||c - d||$, and $\theta(a - b) = (c - d)$ for some $\theta$, will be counted at most $|O(d - 1, \mathbb{F}_q)|$ times in the sum $\sum_{\theta, z} |w_\theta(z)|^2$. If $a = b$ and $c = d$, then the quadruples $(a, b, c, d)$ will be counted at most $|O(d, \mathbb{F}_q)|$ times in the sum $\sum_{\theta, z} |w_\theta(z)|^2$.

Let

$$W := \{(a, b, c, d) \in \mathcal{E}^4 : ||a - b|| = ||c - d||\}.$$ 

It is clear that $\sum_{t \in \mathbb{F}_q} \nu_{\mathcal{E}}(t)^2 = |W|$. If $(a, b)$ and $(c, d)$ belong to $w_\theta(z)$ for some $\theta \in O(d, \mathbb{F}_q)$ and $z \in \mathbb{F}_q^d$, then $(a, b, c, d) \in W$ and $(b, a, d, c) \in W$. From this observation, we get the following

$$\sum_{\theta, z} |w_\theta(z)|^2 \leq \frac{|O(d - 1, \mathbb{F}_q)| |W|}{2} + |O(d, \mathbb{F}_q)| |\mathcal{E}|^2$$

$$\leq \frac{|O(d - 1, \mathbb{F}_q)|}{2} \sum_{t \in \mathbb{F}_q} \nu_{\mathcal{E}}(t)^2 + |O(d, \mathbb{F}_q)| |\mathcal{E}|^2,$$ \hspace{1cm} (3.4)
where the factor $|\mathcal{E}|^2$ comes from the number of quadruples $(a, b, c, d)$ with $a = b$ and $c = d$.

Lemma 3.1 together with the inequalities

$$|O(d - 1, \mathbb{F}_q)| |\mathcal{E}|^4 \leq \frac{|O(d, \mathbb{F}_q)| |\mathcal{E}|^4}{2q}$$

and

$$|O(d, \mathbb{F}_q)||\mathcal{E}|^2 \leq |O(d - 1, \mathbb{F}_q)| q^{d-1}|\mathcal{E}|^2|A_d|$$

leads to

$$\sum_{\theta, z} |w_{\theta}(z)|^2 \leq \frac{|O(d - 1, \mathbb{F}_q)|}{2} \left( \frac{|\mathcal{E}|^4}{q} + 2q^{d-1}|\mathcal{E}|^2|A_d| \right) + |O(d, \mathbb{F}_q)||\mathcal{E}|^2$$

$$\leq 4|O(d - 1, \mathbb{F}_q)| q^{d-1}|\mathcal{E}|^2|A_d|,$$

Combining (3.2) with (3.3), we get

$$\sum_{\theta \in O(d, \mathbb{F}_q), z \in \mathbb{F}_q^d} |w_{\theta}(z)|^{k+1} \leq |O(d, \mathbb{F}_q)| \left( \frac{|\mathcal{E}|^{2k+2}}{q^kd} + 2k(k - 1)|\mathcal{E}|^{k-1}|O(d - 1, \mathbb{F}_q)|(q^{d-1}|\mathcal{E}|^2|A_d|) \right)$$

$$\leq |O(d, \mathbb{F}_q)| \left( \frac{|\mathcal{E}|^{2k+2}}{q^kd} + 2k(k - 1)q^{d-1}|\mathcal{E}|^{k+1}|A_d||O(d - 1, \mathbb{F}_q)|. \right)$$

(3.5)

It follows from (3.1) and (3.5) that

$$\sum_{D} s(\mathbb{D}) \mu(\mathbb{D})^2 \leq |O(d, \mathbb{F}_q)| \left( \frac{|\mathcal{E}|^{2k+2}}{q^kd} + 2k(k - 1)q^{d-1}|\mathcal{E}|^{k+1}|A_d||O(d - 1, \mathbb{F}_q)|. \right)$$

Furthermore we have $s(\mathbb{D}) \leq |O(d - k, \mathbb{F}_q)|$, this implies that

$$\sum_{D} \mu(\mathbb{D})^2 \leq \frac{|\mathcal{E}|^{2k+2}}{q^{k(d-\frac{1}{2})}} + 2k(k - 1)q^{kd-(\frac{k+1}{2})}|\mathcal{E}|^{k+1}|A_d| = (1 + o(1)) \frac{|\mathcal{E}|^{2k+2}}{q^{k(d-\frac{1}{2})}}$$

when $q^{kd} = o \left(|\mathcal{E}|^{k+1}|A_d|^{-1}\right)$. In other words, if $q^{kd} = o \left(|\mathcal{E}|^{k+1}|A_d|^{-1}\right)$, then the number of congruence simplices determined by $\mathcal{E}$ satisfies

$$|T_{k,d}(\mathcal{E})| = (1 - o(1)) q^{(\frac{k+1}{2})},$$

which ends the proof of the theorem. $\Box$

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4 Proofs of Theorems 1.5 and 1.8

If \( q \equiv 1 \mod 4 \) then there are \( 2q - 1 \) points of zero norm in \( \mathbb{F}_q^2 \). In Theorems 1.5 and 1.8, we have \( q = o(|\mathcal{E}|) \), so for the sake of simplicity of this paper, we assume that \( q \equiv 3 \mod 4 \). Before proving Theorems 1.5 and 1.8, we first need the following proposition.

**Proposition 4.1.** Let \( \mathcal{E} \) be a subset in \( \mathbb{F}_q^2 \) and \( \nu_\mathcal{E}(t) \) be the number of pairs \((p, q) \in \mathcal{E} \times \mathcal{E}\) such that \(||p - q|| = t\). We have

\[
\sum_{t \in \mathbb{F}_q^*} \nu_\mathcal{E}(t)^2 \ll \frac{|\mathcal{E}|^4}{q} + q|\mathcal{E}|^{5/2}.
\]

The proof of this proposition is based on the following lemma.

**Lemma 4.2.** Let \( \mathcal{E} \) be a subset in \( \mathbb{F}_q^2 \). For a fixed \( t \in \mathbb{F}_q^* \), denote by \( H_t(\mathcal{E}) \) the number of hinges of the form \((p, q_1, q_2) \in \mathcal{E} \times \mathcal{E} \times \mathcal{E}\) with \(||p - q_1|| = ||p - q_2|| = t\) and \( q_1 \neq q_2 \). We have the following estimate

\[
\sum_{t \in \mathbb{F}_q^*} \nu_\mathcal{E}(t)^2 \leq \frac{|\mathcal{E}|^{10}}{2} \sum_{t \in \mathbb{F}_q^*} H_t(\mathcal{E}) + |\mathcal{E}|^3. \tag{4.1}
\]

**Proof.** For each point \( p \) in \( \mathcal{E} \), let \( x_p^t \) be the number of points \( q \in \mathcal{E} \) satisfying \(||p - q|| = t\). Then one has

\[
H_t(\mathcal{E}) = \sum_{p \in \mathcal{E}} \left( x_p^t \right). \tag{4.2}
\]

On the other hand, by applying the Cauchy-Schwarz inequality, we obtain

\[
\sum_{t \in \mathbb{F}_q^*} \nu_\mathcal{E}(t)^2 = \frac{1}{4} \sum_{t \in \mathbb{F}_q^*} \left( \sum_{p \in \mathcal{E}} x_p^t \right)^2 \leq \frac{1}{4} \sum_{t \in \mathbb{F}_q^*} |\mathcal{E}| \sum_{p \in \mathcal{E}} (x_p^t)^2 \\
\leq \frac{1}{2} \sum_{t \in \mathbb{F}_q^*} |\mathcal{E}| \sum_{p \in \mathcal{E}} \left( x_p^t \right)^2 + \frac{1}{4} \sum_{t \in \mathbb{F}_q^*} |\mathcal{E}| \sum_{p \in \mathcal{E}} x_p^t \\
\leq \frac{1}{2} |\mathcal{E}| \sum_{t \in \mathbb{F}_q^*} H_t(\mathcal{E}) + |\mathcal{E}|^3.
\]

which completes the proof of the lemma. \( \square \)

A reflection about a point \( u \in \mathbb{F}_q^2 \) is a map of the form

\[
R_u(x) = R(x - u) + u,
\]
where \( R \) is a matrix of the form

\[
R = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad a, b \in \mathbb{F}_q, \quad a^2 + b^2 = 1.
\]
For $t \in \mathbb{F}_q^*$, the reflection graph $RF_t(\mathbb{F}_q^2)$ is constructed as

$$V(RF_t(\mathbb{F}_q^2)) = \{ (x, y) \in \mathbb{F}_q^2 \times \mathbb{F}_q^2 : ||x - y|| = t \},$$

and

$$E(RF_t(\mathbb{F}_q^2)) = \{ ((x, y), (z, w)) \in V(RF_t(\mathbb{F}_q^2)) \times V(RF_t(\mathbb{F}_q^2)) : \exists R_u, R_u(x) = z, R_u(y) = w \}.$$

Hanson et al. [7] established the $(n, d, \lambda)$ form of this graph as follows.

**Lemma 4.3** (Hanson et al., [7]). The reflection graph $RF_t(\mathbb{F}_q^2)$ is

$$(q^2(q \pm 1), q^2 \pm q, 2(q \pm 1)) - \text{graph.}$$

The proof of Proposition 4.1 below is quite similar to that of [7, Theorem 1 and Lemma 14], but here we give a straight proof and avoid using the bound $\sum_{t \in \mathbb{F}_q^*} \nu(t)^2 \ll |E|^3 + q^2|E|^2$ which can be proved by employing the distribution of edges between two sets of vertices in the Erdős-Rényi graph with an appropriated setting.

**Proof of Proposition 4.1.** For any two distinct points $q_1$ and $q_2$ in $\mathbb{F}_q^2$, the bisector line $l_{q_1,q_2}$ is defined as

$$l_{q_1,q_2} := \{ x \in \mathbb{F}_q^2 : ||x - q_1|| = ||x - q_2|| \}.$$

Let $\mathcal{L}$ be the multi-set of bisector lines defined as

$$\mathcal{L} = \bigcup_{(q_1,q_2) \in E^2} l_{q_1,q_2}.$$

One can identify each point $(a, b) \in E$ with a vertex $[a, b, 1]$ in the Erdős-Rényi graph $PG(q, 3)$, and each line of the form $cx + dy + e = 0$ in $\mathcal{L}$ with a vertex $[c, d, e]$ in the Erdős-Rényi graph $\mathcal{E}\mathcal{R}(\mathbb{F}_q^3)$. We denote the corresponding sets of vertices in the Erdős-Rényi graph $\mathcal{E}\mathcal{R}(\mathbb{F}_q^3)$ by $\mathcal{E}'$ and $\mathcal{L}'$, respectively.

Hence, we have $|\mathcal{E}'| = |E|$, $|\mathcal{L}'| = |\mathcal{L}|$, and the sum $\sum_{t \in \mathbb{F}_q^*} H_t(\mathcal{E})$ is equal to the number of edges between $\mathcal{E}'$ and $\mathcal{L}'$ in $\mathcal{E}\mathcal{R}(\mathbb{F}_q^3)$. Therefore, by Lemmas 2.1 and 2.2 we get

$$\sum_{t \in \mathbb{F}_q^*} H_t(\mathcal{E}) = I(\mathcal{E}', \mathcal{L}') \leq \frac{|\mathcal{E}||\mathcal{L}|}{q} + q^{1/2}|\mathcal{L}|^{1/2} \sqrt{\sum_{l \in \mathcal{L}} w(l)^2},$$

where $w(l)$ is the multiplicity of $l \in \mathcal{L}$.

If $l_{x,z} = l_{y,w}$ and $||x - z|| \neq 0$, then one can check that there exists a unique reflection $R_u$ such that $R_u(x) = z$, $R_u(y) = w$, and $||x - y|| = ||z - w||$. Thus the sum $\sum_{l \in \mathcal{L}} w(l)^2$ is the cardinality of the following set

$$Q := Q(\mathcal{E}) = \{ (x, y, z, w) \in \mathcal{E}^4 : \exists t \in \mathbb{F}_q^* \ ( (x, y), (z, w) ) \in E(RF_t(\mathbb{F}_q^2)) \}.$$
We set
\[ Q_t := \{(x, y, z, w) \in Q : ||x - y|| = ||z - w|| = t\}, \]
and see that
\[ \sum_{l \in L} w(l)^2 = \sum_{t \in \mathbb{F}_q^*} |Q_t|. \]
For each \( t \neq 0 \), it follows from Lemma 4.3 and Lemma 2.1 that
\[ |Q_t| \leq \frac{\nu_E(t)^2}{q} + 2(q - 1)\nu_E(t). \]
Hence, we get
\[ \sum_{l \in L} w(l)^2 \leq \frac{\sum_{t \in \mathbb{F}_q^*} \nu_E(t)^2}{q} + 2(q - 1)|E|^2. \]
Combining (4.1) with (4.2), we obtain
\[ \sum_{t \in \mathbb{F}_q^*} \nu_E(t)^2 \leq |E| \left( \frac{|E||L|}{q} + q^{1/2}|E|^{1/2} \sqrt{\frac{\sum_{t \in \mathbb{F}_q^*} \nu_E(t)^2}{q} + 2(q - 1)|E|^2} \right) + |E|^3. \]
Solving this inequality leads to the desired bound, and the proposition follows.

**Remark 4.4.** It follows from the proof of Proposition 4.1 that the number of the number of hinges determined by points in \( E \) is at most
\[ \frac{|E|^3}{q} + q^{1/2}|E|^{1/2} \sqrt{\frac{\sum_{t \in \mathbb{F}_q^*} \nu_E(t)^2}{q} + 2(q - 1)|E|^2}. \]
On the other hand, we have proved that
\[ \sum_{t \in \mathbb{F}_q^*} \nu_E(t)^2 \ll \frac{|E|^4}{q} + q|E|^{5/2}. \]
This leads to that the number of hinges is at most \( \ll |E|^3/q \) when \( |E| \gg q^{4/3} \).

As an application of Proposition 4.1, we obtain the following result.

**Theorem 4.5 (Bennett et al. [3]).** Let \( E \) be a set of points in \( \mathbb{F}_q^2 \). If \( |E| \gg q^{8/5} \), then \( E \)
determines a positive proportion of all triangles.

**Proof.** The proof of Theorem 1.5 is very similar to that of Theorem 1.2, and there is the only one different step. That is, instead of using Lemma 3.1, we use Proposition 4.1, thus
we leave the rest to the reader. \( \square \)

In order to prove Theorem 1.8, we make use of the following theorem on the cardinality of a maximal independent set of a hypergraph due to Spencer [9].
Theorem 4.6. Let $H$ be a $k$-uniform hypergraph of $n$ vertices and $m$ edges with $m \geq n/k$, and let $\alpha(H)$ denote the independence number of $H$. Then

$$\alpha(H) \geq \left(1 - \frac{1}{k}\right) \left(\frac{n^k}{km}\right)^{\frac{1}{k-1}}.$$ 

**Proof of Theorem 1.8.** We call a 4-tuple of distinct elements in $E^4$ regular if all six generalized distances determined are distinct. Otherwise, it is called singular. Let $H$ be the 4-uniform hypergraph on the vertex set $V(H) = \mathcal{E}$, whose edges are the singular 4-tuples of $\mathcal{E}$.

On one hand, it follows from the remark (4.4) that the number of 4-tuples containing a triple induced a hinge is at most $((1 + o(1))|\mathcal{E}|^3/q) \cdot |\mathcal{E}| = (1 + o(1))|\mathcal{E}|^4/q$ when $|\mathcal{E}| \gg q^{4/3}$. Thus, the number of edges of $H$ containing a triple induced a hinge is at most $(1 + o(1))|\mathcal{E}|^4/q$.

On the other hand, according to Proposition 4.1 and from the fact that the number of quadruples with zero-distances is no more than $4q^4$, the number of edges of $H$ that do not contain any hinge is at most $(1 + o(1))|\mathcal{E}|^4/q$, when $|\mathcal{E}| \gg q^{4/3}$.

In other words, if $|\mathcal{E}| \gg q^{4/3}$, we have

$$|E(H)| \leq (1 + o(1))\frac{|\mathcal{E}|^4}{q}.$$ 

By Theorem 4.6, one has

$$\alpha(H) \geq C \left(\frac{|\mathcal{E}|^4}{|E(H)|}\right)^{1/3} = Cq^{1/3},$$

for some positive constant $C$. Since there is no repeated distance determined by the independent set of $H$, there exists a distinct distance subset $U \subseteq \mathcal{E}$ satisfying $|U| \geq \alpha(H) \geq Cq^{1/3}$.

Moreover, it is easy to see that there is at least one repeated distance determined by any set of $\sqrt{2q}^{1/2} + 1$ elements since there are only $q = |F_q|$ distances over $\mathbb{F}_q$. This concludes the proof of the theorem.

5 Discussions

As mentioned in the introduction, one of the most important steps in their proof in [3] is to reduce the problem of counting congruence classes of $k$-simplices to the number of quadruples $(a, b, c, d) \in \mathcal{E}^4$ with $||a - b|| = ||c - d||$ by applying Lemma 2.3 and elementary results from group action theory in an ingenious way. In this section we will explain more precise the role of the error term $q^d |\mathcal{E}|^2$ in the proof of Theorem 1.1 in [3], which is an important advantage of the method in [3]. It follows from the proof of Theorem 1.2 that

$$|T_{k,d}(\mathcal{E})| \geq \frac{(\sum_D \mu(D))^2}{\sum_D \mu(D)^2} \geq \frac{|\mathcal{E}|^{2k+2}}{\sum_D \mu(D)^2},$$

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and
\[ \sum_{D} s(D) \mu(D)^2 \leq \sum_{\theta \in O(d,F_q), z \in F_q^d} |w_\theta(z)|^{k+1}. \]

We now present two approaches to bound \( \sum_{D} s(D) \mu(D)^2 \).

**The first approach:** We use an observation that \( |w_\theta(z)|^{k+1} \ll |E|^{k-1} |w_\theta(z)|^2 \), which implies that
\[ \sum_{D} s(D) \mu(D)^2 \leq \sum_{\theta \in O(d,F_q), z \in F_q^d} |w_\theta(z)|^{k+1} \ll |E|^{k-1} \sum_{\theta \in O(d,F_q), z \in F_q^d} |w_\theta(z)|^2. \tag{5.1} \]

Moreover, from the proof of Theorem 1.2, we also have
\[
\sum_{\theta, z} |w_\theta(z)|^2 \leq \frac{|O(d-1,F_q)||W|}{2} + |O(d,F_q)||E|^2 \\
\leq \frac{|O(d-1,F_q)|}{2} \left( \sum_{t \in F_q} \nu_{E}(t)^2 \right) + |O(d,F_q)||E|^2 \\
\leq \frac{|O(d-1,F_q)|}{2} \left( \frac{|E|^4}{q} + q^d |E|^2 \right) + |O(d,F_q)||E|^2,
\]
where we use
\[ \sum_{t \in F_q} \nu_{E}(t)^2 \leq \frac{|E|^4}{q} + q^d |E|^2. \]

Suppose that \( |E| \gg q^{d-\frac{d}{k+1}} \), we have \( \sum_{t \in F_q} \nu_{E}(t)^2 \leq (1 + o(1)) \frac{|E|^4}{q} \). This implies that
\[ \sum_{\theta, z} |w_\theta(z)|^2 \leq \frac{|O(d-1,F_q)||W|}{2} + |O(d,F_q)||E|^2 \\
\leq \frac{|O(d-1,F_q)|}{2} \left( (1 + o(1)) \frac{|E|^4}{q} \right) + |O(d,F_q)||E|^2, \tag{5.2} \]

Putting (5.1) and (5.2) together gives us
\[ \sum_{D} s(D) \mu(D)^2 \leq \sum_{\theta \in O(d,F_q), z \in F_q^d} |w_\theta(z)|^{k+1} \ll |E|^{k-1} \frac{|O(d-1,F_q)|}{2} \left( \frac{|E|^4}{q} \right). \tag{5.3} \]

**The second approach:** If we apply Lemma 2.3, then as in (3.2) in the proof of Theorem 1.2, we are able to get rid off the bigger term \( |E|^4/q \) and obtain the following:
\[ \sum_{D} s(D) \mu(D)^2 \leq \sum_{\theta \in O(d,F_q), z \in F_q^d} |w_\theta(z)|^{k+1} \ll |O(d,F_q)| \frac{|E|^{2k+2}}{q^{kd}} + |E|^{k-1} \frac{|O(d-1,F_q)|}{2} (q^d |E|^2), \]

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which is better than the estimate (5.3). Therefore, we obtain a better threshold for the condition on the size of $\mathcal{E}$ in Theorem 1.1.

In other words, the method in [3] is a very strong and effective tool to deal with combinatorial geometry problems. We believe that there are many problems which can be handled by the same approach.

In the rest of this section, we mention a connection between Fourier analytic methods and methods from spectral graph theory. Many results obtained by Fourier analytic methods can be proved by using our techniques and vice versa. There are some problems in which we can see a step-by-step correspondence between these methods, for instance, in [14] the third listed author presented a correspondence on counting the number of pairs of points $(x, y) \in \mathcal{E} \times \mathcal{E}$ with $||x - y|| = 1$. We refer the reader to [14] for a detailed explanation. In general, it is not easy to see the relation between these methods. For example, we now discuss some proofs for bounding the sum $\sum_{\theta, z} |w_\theta(z)|^2$.

The authors of [3] gave an upper bound for the sum $\sum_{\theta, z} |w_\theta(z)|^2$ by using directly the Fourier transform of the functions $|w_\theta(z)|$ and properties of characters. However, in this way, we do not know so much about geometric meanings of the problem. In order to handle the problem by our methods, we need to start from geometric properties of $w_\theta(z)$ to find an appropriated setting in a certain graph. More precisely, if $(u, v)$ and $(x, y)$ are pairs in $w_\theta(z)$, then we have

$$\theta(u) + z = v, \theta(x) + z = y.$$ 

This implies that

$$\theta(u - x) = v - y, \ |u - x| = |v - y|.$$ 

Hence, if we have $|u - x| = |v - y|$ and $u \neq x$, then there exist $k$ pairs $(\theta, z)$ such that $(u, v)$ and $(x, y)$ belong to $w_\theta(z)$, where $k$ is the number of stabilizers of $u - x$. Thus, the sum $\sum_{\theta, z} |w_\theta(z)|^2$ can be estimated by using the upper bound of $\sum_{t \in \mathbb{F}_q} \nu_\mathcal{E}(t)^2$ and the size of stabilizers of a non-zero point in $\mathbb{F}_q^d$. We now present two proofs of an upper bound of $\sum_{t} \nu_\mathcal{E}(t)^2$ by methods from spectral graph theory. In particular, we prove the following

$$\sum_{t \in \mathbb{F}_q} \nu_\mathcal{E}(t)^2 \leq \frac{|\mathcal{E}|^4}{q} + q^d|\mathcal{E}|^2.$$ 

**The first proof:** We define two sets $\mathcal{B}$ and $\mathcal{C}$ in the Erdős-Rényi graph $\mathcal{E}R(\mathbb{F}_q^{2d+2})$ as follows:

$$\mathcal{B} = \{ [-2x, 2u, 1, ||x|| - ||u||] : x, u \in \mathcal{E} \} \subseteq \mathbb{F}_q^{2d+2},$$

$$\mathcal{C} = \{ [y, v, ||y|| - ||v||, 1] : y, v \in \mathcal{E} \} \subseteq \mathbb{F}_q^{2d+2}.$$ 

One can check that if $||x - y|| = ||u - v||$, then there is an edge between

$$[-2x, 2u, 1, ||x|| - ||u||] \in \mathcal{B}$$

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and
\[ [y, v, \|y\| - \|v\|, 1] \in C \]
in the Erdős-Rényi graph \( \mathcal{E} R(\mathbb{F}_q^{2d+2}) \). This implies that \( \sum_t \nu_t(t)^2 \) is equal to the number of edges between \( B \) and \( C \). It follows from Lemmas 2.1 and 2.2 that
\[ e(B, C) \leq |E|^4 q + q^d |E|^2. \]
This completes the proof of the result.

**The second proof:** It has been shown in [14] that for \( E \subseteq \mathbb{F}_q^d \) and a fixed \( t \in \mathbb{F}_q \setminus \{0\} \), the number of pairs \( (x, y) \in E \times E \) with \( \|x - y\| = t \) is at most \( \frac{|E|^2}{q} + q^{d-1/2} |E| \). This leads to that the number of quadruples \( (u, v, x, y) \in E^4 \) with \( \|u - v\| = \|x - y\| = t \) is at most \( \ll |E|^4 / q^2 + q^{d-1} |E|^2 \). Thus, we obtain
\[ \sum_{t \in \mathbb{F}_q} \nu_t(t)^2 \ll \frac{|E|^4}{q} + q^d |E|^2. \]

By using the same techniques as in the first proof, one can prove that the number of pairs \( (u, v) \in E \times E \) with \( \|u - v\| = 0 \) is at most \( \frac{|E|^2}{q} + q^{d/2} |E| \). Therefore, the number of quadruples \( (u, v, x, y) \in E^4 \) with \( \|u - v\| = \|x - y\| = 0 \) is at most \( \ll \frac{|E|^4}{q^2} + q^d |E|^2 \). In conclusion, we have
\[ \sum_{t \in \mathbb{F}_q} \nu_t(t)^2 \ll \frac{|E|^4}{q} + q^d |E|^2. \]

In conclusion, in general, there is a correspondence between Fourier analytic methods and techniques from spectral graph theory. The ideas for handling problems by graph-theoretic techniques usually come from the geometric properties of the problems. Many results obtained from the Fourier methods are hard to derive from the graph theory method. The spectral methods sometimes give us many simple applications without invoking more advanced tools from analytic number theory.

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References


University of Education, 
Vietnam National University 
E-mail: phamduchiepk6@gmail.com

Department of Mathematics, 
EPF Lausanne 
Switzerland 
E-mail: thang.pham@epfl.ch

University of Education, 
Vietnam National University 
Viet Nam 
E-mail: vinhla@vnu.edu.vn